Risk-Limiting Dispatch for Integrating Renewable Power*

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Abstract

Risk-limiting dispatch or RLD is formulated as the optimal solution to a multi-stage, stochastic decision problem. At each stage, the system operator (SO) purchases forward energy and reserve capacity over a block or interval of time. The blocks get shorter as operations approach real time. Each decision is based on the most recent available information, including demand, renewable power, weather forecasts. The accumulated energy blocks must at each t match the net demand $D(t) = L(t) - W(t)$. The load L and renewable power W are both random processes. The expected cost of a dispatch is the sum of the costs of the energy and reserve capacity and the penalty or risk from mismatch between net demand and energy supply. The paper derives computable 'closed-form' formulas for RLD. Numerical examples demonstrate that the minimum expected cost can be substantially reduced by recognizing that risk from current decisions can be mitigated by future decisions; by additional intra-day energy and reserve capacity markets; and by better forecasts. These reductions are quantified and can be used to explore changes in the SO's decision structure, forecasting technology, and renewable penetration.

1 Introduction

States are setting ambitious goals for electricity from clean energy. California's goal is 33% by 2020. The goals are promoted by 'renewable portfolio standards' or RPS, which require electricity suppliers to produce a specified fraction of their electricity from renewables. The system operator (SO) has the task of integrating growing amounts of renewable power into the power grid.

The SO makes a sequence of decisions to balance the supply and load of electric power at every instant, in the face of several unknowns: future load is uncertain; renewable generation is highly variable and unpredictable; prices fluctuate. Forecasts of these variables have errors. The forecasts get more accurate as observations accumulate with the approach of real time – when supply and load must be balanced.

Based on the forecasts, the SO purchases blocks of conventional power ahead of real time. The purchase contracts guarantee delivery of power over a specified block or interval of time, with each block getting shorter as operations approach real time. These blocks are accumulated by the SO, and "stacked" in a manner of speaking, one upon another, so that the stack of conventional (dirty) power plus the renewable (clean) power closely matches the real-time load. For example, in a four-step market process, the transactions might be staggered as follows (see Figure 1):

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- 24 hour-ahead, SO purchases s_1 for a 1 hour block;
- 1 hour-ahead, SO purchases s_2 for a 30 minute block;
- 15 minutes-ahead, SO purchases s_3 for a 5 minute block;
- 5 minutes-ahead, SO purchases s_4 for a 1 minute block.

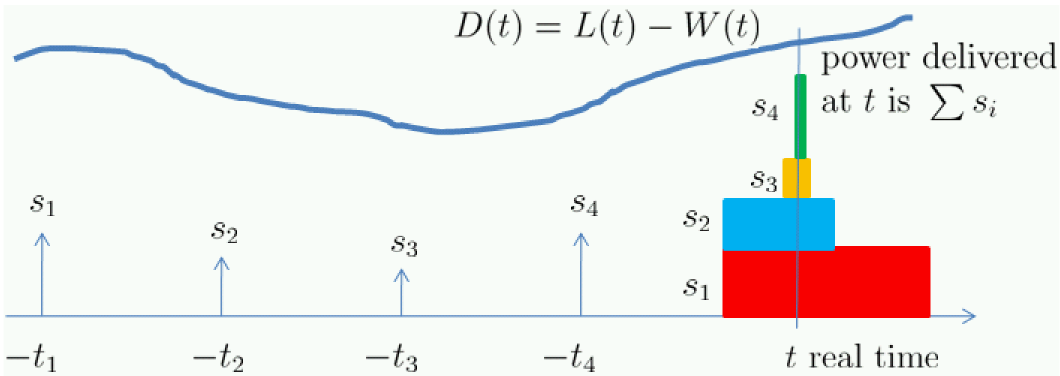

Figure 1: Power block s_i is purchased t_i-ahead of real time t; net demand is $D(t)$.

In addition to these forward or future energy purchases, the SO hedges against forecast errors and equipment failure by purchasing *reserve capacity*. A reserve capacity contract is a call option: upon paying the option price, the SO secures the right but not the obligation to purchase energy in a particular forward market. We model such a reserve capacity contract as a two-part transaction comprising the purchase of a power block in a forward market followed by a sale of that power if the reserve capacity is not utilized. (The difference between the purchase and sale price is the price of the call option or the reserve capacity.) With this model there is no need to distinguish between forward energy and capacity contracts, and the SO's decisions comprises a sequence of forward energy buy or sell transactions. The sequence of transactions is called the *dispatch*.

This paper introduces *Risk Limiting Dispatch* or RLD. RLD is a formulation of the SO's optimal dispatch – the decisions that minimize the expected cost while ensuring adequate reliability. Let $W(t)$ be the wind power available to the SO. Let $L(t)$ be the load. Both are stochastic processes. The difference $D(t) = L(t) - W(t)$ is the *net demand*. $D(t)$ must be made up by the SO's market transactions. There are m types of markets, distinguished by the blocks or intervals (1 hour, 30 min, and so on) and how far ahead the block must be purchased (24 hour-ahead, 1 hour-ahead, and so on). The cost of each power block is known in advance.

At each t, the SO receives a measurement vector $y(t)$ which provides information about $W(t)$ and $L(t)$. The measurements may include current load and renewable power generation, and weather forecasts. A dispatch policy specifies the market transactions to be undertaken at each t as a function of the available observations $Y(t) = \{y(\tau),\ \tau \leq t\}$. If $S(t)$ is the power accumulated by the SO at real time t and the net demand is $D(t)$ there is a penalty (which may be infinite) if $S(t) \neq D(t)$; the penalty measures operational risk. The RLD or optimal dispatch policy minimizes the expected cost of the market transactions plus the penalty. Following standard decision theory, RLD is formulated as the solution to a m-stage optimal stochastic control problem.

Contribution of the paper

We obtain a closed form solution to the RLD when the prices of power blocks are constant. (A computationally more difficult closed form is obtained when the prices are not constant.) The random load, renewable power and forecast error processes have arbitrary probability distributions. RLD has an appealing form. Suppose the net cumulative purchases in the first $(k-1)$ markets is x_{k-1}. With the information available just before the decision in the kth market is to be made, RLD computes two thresholds, $\varphi_k^+ < \varphi_k^-$, using a specified formula. In the kth market, if $x_{k-1} < \varphi_k^+$, RLD purchases the additional amount $s_k = \varphi_k^+ - x_{k-1}$. If $x_{k-1} > \varphi_k^-$, RLD sells the excess $s_k = x_{k-1} - \varphi_k^-$. If $\varphi_k^+ < x_{k-1} < \varphi_{k-1}^-$, RLD makes no purchase or sale. If we interpret the interval $[\varphi_k^+, \varphi_k^-]$ as the *target*, the SO's optimum decision is to use the available information to calculate this target and then make the smallest purchase or sale to bring the accumulated power into the target interval.

It is commonly assumed that forecast errors are Gaussian. The thresholds then take the very simple form:

$$\varphi_k^+ = \mu_k + \Delta_k^+, \;\; \varphi_k^- = \mu_k + \Delta_k^-. \tag{1}$$

In formula (1), μ_k is the forecast of the net demand $D(t)$ conditioned on the available information, and Δ_k^+, Δ_k^- are pre-computable constants, which we call *risk premiums*. If the forecast has no error, the risk premiums are zero, the target interval shrinks to a point $\{\mu_k\}$, and the SO's optimal decision is to accumulate supply equal to the net demand forecast.

Usually of course there is a forecast error; Δ_k^+, Δ_k^- are then non-zero and can be regarded as the (optimal) hedge against the risk or penalty of not matching the net demand. It may come as a surprise that when the SO has future opportunities to change the accumulated power supply ($k < m$), the risk premium is frequently *negative*. The paper presents several numerical examples to show that RLD has substantially lower cost than current dispatch procedures, outlined below. More importantly, the RLD closed-form solution provides a quantitative evaluation of the benefits of additional intra-day markets and better information. Such quantitative results can inform the design of dispatch processes to integrate large amounts of renewables.

RLD was first formulated as a stochastic optimal control problem in [14], which gives the important property of convexity of the value function and the first-order optimality conditions. Our study extends [14] in several respects. First, we consider blocks of different duration, whereas [14] only permits a single duration, so that (for example) the distinction between 24-hour ahead one-hour blocks and 15-min ahead 5-min load-following blocks cannot be made. Also, [14] does not permit sale of energy, so reserve capacity cannot be considered. The important RLD threshold property (when marginal costs are constant) and the simple form (1) are not derived in [14], so the numerical examples we present would be very difficult to obtain. The present paper generalized the 2-step RLD in [16].

Current practice

In practice the SO does not follow RLD. Instead for each market $j = 1, \cdots, m$, the SO makes the decision that minimizes the immediate cost, based on the forecast, but neglecting the fact that future decisions based on more accurate forecasts, will correct current decisions. Thus the current procedure is *decoupled dispatch*, since each decision is determined independently of the others. Decision-theoretic considerations immediately imply that decoupled dispatch must incur a larger cost than the optimal dispatch RLD. The question is: how large is the 'cost gap'? Our numerical examples suggest that the gap is large.

California's Independent System Operator (CAISO) has commissioned several studies to simulate its current dispatch process to determine how much more reserve power will be needed to meet future 20 and 33 percent RPS targets. (CAISO market operations are described in [3].) The basic simulation approach, developed in [11], proceeds as follows. One takes a recorded trace of net demand for several days. Several forecast error traces are then produced as Monte Carlo runs of a Gaussian process. The sum of a net demand trace and an error trace is taken as a sample path of the net demand forecast, which is used to calculate the dispatch.

In the first stage 24 hour-ahead of real time, CAISO schedules one-hour blocks of power to match the hourly demand forecast. Two hours ahead, the next hourly scheduled block is adjusted (up or down) to match CAISO's two hour-ahead forecast. In the second, real-time or balancing stage, 5-min blocks of power –

called load following – are scheduled 15 min-ahead to compensate for the difference between the 15 min-ahead forecast and the adjusted one-hour blocks. Lastly, there is a second-by-second purchase of energy – called regulation power – to automatically compensate for frequency deviations resulting from demand-supply mismatch. The resulting simulation gives the load-following and regulation power. These are random quantities, which depend on the randomly generated forecast sample paths. The random quantities are revealingly visualized in [7].

In order to simulate the impact of 20 or 33 percent RPS, the same procedure is followed [2, 4]. The only modification is that one magnifies recorded traces of renewable power to create traces that meet the RPS requirement. These studies find that large reserves are needed to absorb the variability of renewable power. For example, [9] concludes that "the amount of regulation and imbalance [load-following] energy dispatched in real time [. . .] to maintain system performance within acceptable limits during morning and evening ramp hours for 33 percent renewable cases in 2020 was 4,800 MW." Such fast-acting energy is costly, and may make meeting California's 2020 goal of 33 percent renewable energy financially unsupportable. (The study admits that the 4,800 MW requirement may be "optimistic, in that the impact of large forecast errors for renewable production, especially forecast errors associated with wind production, were not studied.")

The simulation studies show that most of the reserve requirements stem from the current CAISO practice of stacking only two kinds (one-hour and 15-min) of power blocks. From Figure 1 one can see that if renewable power (part of $D(t)$ in the figure) exhibits large intra-hour variations (ramps), the burden of compensating for such variations will fall on the 5-min load-following blocks and the balancing energy needs will be high. However, if CAISO were also to schedule shorter blocks (30-min, 15-min, etc.) at shorter times ahead – and hence with lower forecast errors – the reserve requirements would be lower. The increased flexibility gained by such intra-day markets is advocated by NERC's Task Force on Integrating Variable Generation [8].

Limitations of the paper

The RLD presented here and the above-referenced simulation studies have three limitations. First, they ignore transmission constraints and losses. Second, they do not consider unit commitment (UC). UC introduces integer-valued decision variables which increase the computational burden and preclude mathematical analysis. Two studies [1, 13] include UC start-up costs, but computational complexity restricts them to a 2-stage dispatch, evaluated only via numerical examples. So these studies cannot predict the reduction in needed reserves from intra-day markets or better information. Note, moreover, that once the unit commitment decision is made, RLD indeed determines the optimal power block purchases.

The rest of the paper is organized as follows. The dispatch procedure is formulated as an m-stage stochastic control problem in §2. The RLD optimality conditions are derived in §3. The special case of Gaussian forecast errors is treated in §4. Two examples occupy §5. Conclusions are summarized in §6. Proofs are collected in the Appendix.

2 Model of dispatch

The dispatch model specifies the SO's energy supply, and formulates a dispatch policy and its cost.

2.1 Energy supply

Net load or demand is denoted by $D(t) = L(t) - W(t),\ t \geq 0$, with $L(t)$ and $W(t)$ being the true load and wind power. L and W are stochastic processes with arbitrary probability distributions. Time t is discrete, measured in hours. The dispatch constructs a supply $S(t)$, $t \geq 0$, by stacking up blocks of power purchased in m markets. Blocks in the kth market have duration T_k, with $T_1 \geq \cdots \geq T_m$. We assume that T_{j-1} is a multiple of T_j, $T_{j-1} = N_j T_j$. These are the SO's market transactions:

- SO buys (or sells) T_1 blocks of magnitude $s(k_1)$ for the interval $[k_1 T_1,\ (k_1+1)T_1]$, $k_1 = 0, \cdots, N_1 - 1$;
- For each k_1, SO buys (or sells) T_2 blocks of magnitude $s(k_1, k_2)$ for the interval $[k_1 T_1 + k_2, T_2,\ k_1 T_1 + (k_2+1)T_2]$, $k_2 = 0, \cdots, N_2 - 1$; ...
- For each $k_1, \cdots, k_{m-1}$, SO buys (or sells) T_m blocks of magnitude $s(k_1, \cdots, k_m)$ for the interval $[k_1 T_1 + \cdots + k_m T_m,\ k_1 T_1 + \cdots + (k_m+1)T_m]$.

By convention, $s(k_1, \cdots, k_j) > 0$ or < 0 accordingly as a power block is purchased or sold. The power blocks of Figure 2 are stacked to form the energy supply function over the time horizon $[0, N_1 T_1]$, which is hours or days long.

The array of power blocks $\{s(k_1, \cdots, k_j)\}$ delivers power $S(t)$ at real time t:

$$S(t) = s(k_1) + s(k_1, k_2) + \cdots + s(k_1, \cdots, k_m),\ t \in [k_1 T_1 + \cdots + k_m T_m,\ k_1 T_1 + \cdots + (k_m+1)T_m]. \tag{2}$$

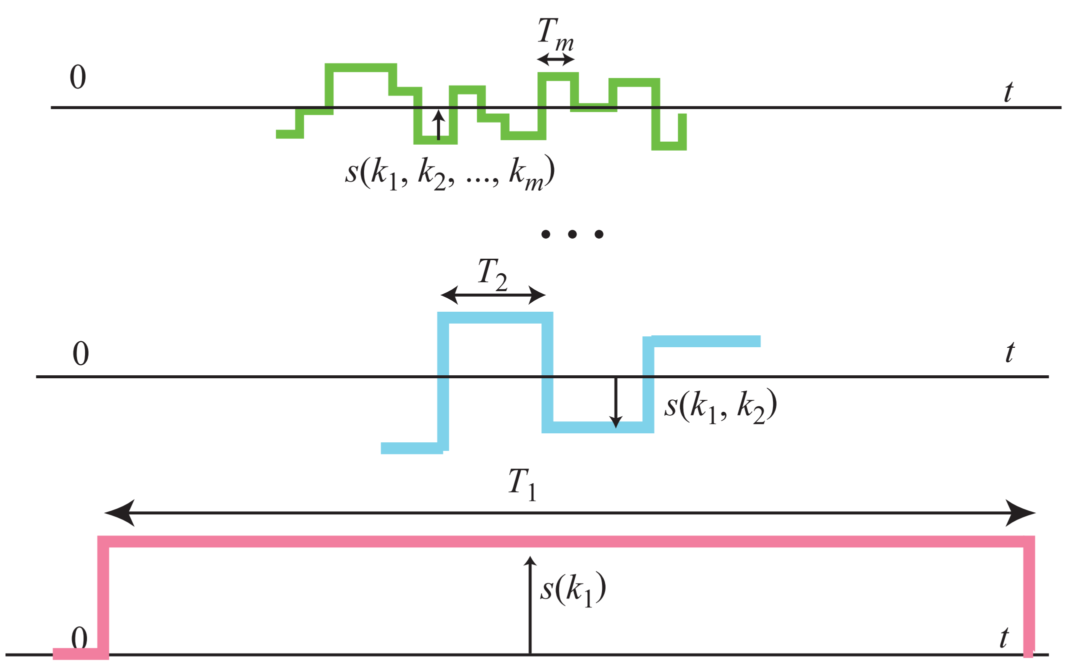

Figure 2: Supply $S(t)$ is constructed by stacking blocks of duration $T_1, \cdots, T_m$.

2.2 Dispatch policy

A dispatch policy selects block $s(k_1, \cdots, k_j)$ at the pre-specified market closing time $t(k_1, \cdots, k_j)$. We assume that for all j

$$t(k_1, \cdots, k_j) \leq k_1 T_1 + \cdots + k_j T_j, \quad (3)$$
$$t(k_1, \cdots, k_j) \leq t(k_1, \cdots, k_j, k_{j+1}). \quad (4)$$

Assumption (3) says that power must be purchased before it is delivered. Typically the market closes a fixed time ahead, e.g., $t(k_1) = k_1 T_1 - 24$ for the 24 hours-ahead energy market. Assumption (4) is an intuitive requirement: block $s(k_1)$ must be purchased before its sub-blocks $s(k_1, k_2)$, which must be purchased before its sub-blocks $s(k_1, k_2, k_3)$, and so on.

We assume that the blocks $s(k_1), \cdots, s(k_1, \cdots, k_m)$ can be selected independently. This assumption is violated if generators impose inter-temporal constraints on successive power blocks. For example, some generators have a minimum power-on duration, and some (e.g. stored hydro) have a total energy constraint.

The net demand $D(t) = L(t) - W(t)$ is random because the load $L(t)$, wind power $W(t)$, or both are random. Forced outages of generation or other equipment are readily modeled as random increases in load as in [5]. At time t an observation $y(t)$ provides information about the net demand. The observations up to t comprise the array $Y(t) = \{y(\tau), \tau \leq t\}$.

The selection of block $s(k_1, \cdots, k_j)$ is based on the available information, so $s(k_1, \cdots, k_j)$ is a function of $Y(t(k_1, \cdots, k_j))$. Formally $s(k_1, \cdots, k_j)$ is adapted to the σ-field $\mathcal{Y}(t(k_1, \cdots, k_j))$ generated by the random variables $Y(t(k_1, \cdots, k_j))$. A **dispatch policy** π for the interval $[0, N_1 T_1]$ is any array of functions,

$$\pi = \{s(k_1, \cdots, k_j) \text{ is adapted to } \mathcal{Y}(t(k_1, \cdots, k_j))\}. \quad (5)$$

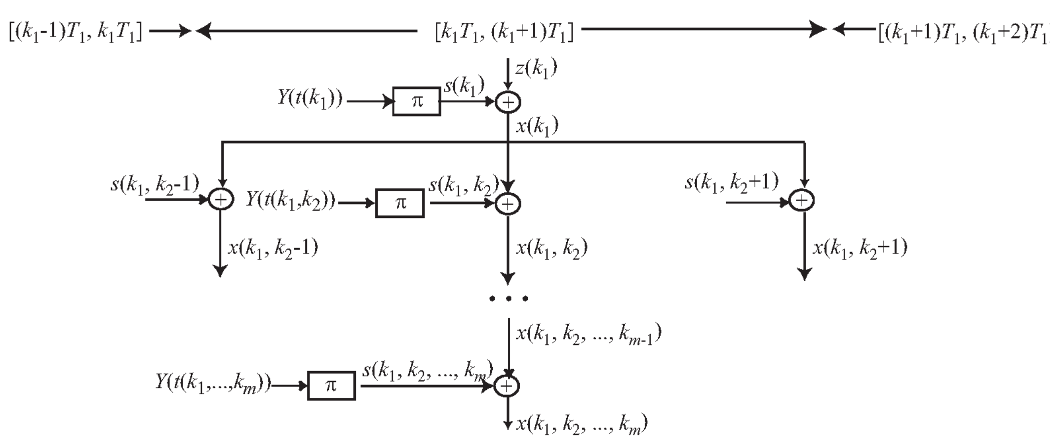

Figure 3: Dispatch policy π selects $s(k_1, \cdots, k_j)$ as a function of $Y(t(k_1, \cdots, k_j))$.

An initial block of power $z(k_1)$ has been secured before time $k_1 T_1$, as the result of actions (such as must-run generation) taken outside of our decision problem. If $z(k_1)$ is random, it must be $Y(t(k_1))$-adapted, so its value is known when $s(k_1)$ is to be selected.

Let $x(k_1, \cdots, k_j)$ be the total power secured just after $s(k_1, \cdots, k_j)$ is selected:

$$x(k_1, \cdots, k_j) = z(k_1) + s(k_1) + s(k_1, k_2) + \cdots + s(k_1, \cdots, k_j). \quad (6)$$

We regard $x(k_1,\cdots,k_j)$ as the **state** and $s(k_1,\cdots,k_j)$ as the **decision** at **stage** $(k_1,\cdots,k_j)$, based on the **observations** $Y(t(k_1,\cdots,k_j))$. The state satisfies the update equation (see Figure 3):

$$x(k_1,\cdots,k_j) = x(k_1,\cdots,k_{j-1}) + s(k_1,\cdots,k_j), \;\; j = 1,\cdots,m. \tag{7}$$

$s(k_1,\cdots,k_j)$ may be positive or negative accordingly as power is purchased or sold. The power eventually delivered at real time $t \in [k_1T_1 + \cdots k_mT_m,\; k_1T_1 + \cdots (k_m+1)T_m]$ is $x(k_1,\cdots,k_m)$.

2.3 Cost and penalty

The cost of block $s = s(k_1,\cdots,k_j)$ is a known convex function $C(k_1,\cdots,k_j;s)$. At first, we consider the case when the marginal cost is constant:

$$C(k_1,\cdots,k_j;s) = \begin{cases} c^+(k_1,\cdots,k_j)\times s, & s = s_+ \geq 0 \\ c^-(k_1,\cdots,k_j)\times s, & s = s_- \leq 0 \end{cases}.$$

$c^+(k_1,\cdots,k_j)$ and $c^-(k_1,\cdots,k_j)$ are unit prices for buying and selling power during $[k_1T_1 + \cdots k_jT_j,\; k_1T_1 + \cdots (k_j+1)T_j]$ so the cost of block $s(k_1,\cdots,k_j)$ is

$$T_j[c^+(k_1,\cdots,k_j)s_+(k_1,\cdots,k_j) + c^-(k_1,\cdots,k_j)s_-(k_1,\cdots,k_j)].$$

The expected cost of a policy π for the interval $[0, N_1T_1]$ is

$$\begin{aligned} J(\pi) &= \mathbb{E}\,\Big\{T_1\sum_{k_1=0}^{N_1-1}[c^+(k_1)s_+(k_1) + c^-(k_1)s_-(k_1)] \\ &+ T_2\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}[c^+(k_1,k_2)s_+(k_1,k_2) + c^-(k_1,k_2)s_-(k_1,k_2)] \\ &+ \cdots \\ &+ T_m\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}\cdots\sum_{k_m=0}^{N_m-1}[c^+(k_1,\cdots,k_m)s_+(k_1,\cdots,k_m) + c^-(k_1,\cdots,k_m)s_-(k_1,\cdots,k_m)] \\ &+ T_m\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}\cdots\sum_{k_m=0}^{N_m-1}[g(d(k_1,\cdots,k_m),\; x(k_1,\cdots,k_m))]\Big\} \end{aligned} \tag{8}$$

In (8), $\mathbb{E}$ denotes mathematical expectation. Except for the last term, the right hand side in (8) is the energy cost for the interval $[0, N_1T_1]$. In the last term in (8) $d(k_1,\cdots,k_m)$ is the net demand, $x(k_1,\cdots,k_m)$ is the power supplied, and $T_m[g(d(k_1,\cdots,k_m),\; x(k_1,\cdots,k_m))]$ is the **risk** or penalty of imbalance during $[k_1T_1 + \cdots + k_mT_m,\; k_1T_1 + \cdots + (k_m+1)T_m]$. The total cost $J(\pi)$ is energy cost plus the imbalance penalty.

Risk-limiting dispatch or RLD is the policy π^* that minimizes the cost $J(\pi)$:

$$\begin{aligned} J(\pi^*) &= \min_{\pi} J(\pi) \\ \text{s.t.} &\quad (5) \end{aligned} \tag{9}$$

RLD π^* is determined in §3 under the following restrictions on the prices $c^+,\; c^-$ and the penalty g.

Restriction on prices In order to avoid trivial cases the prices satisfy

$$c^+(k_1,\cdots,k_m) > c^+(k_1,\cdots,k_{m-1}) > \cdots > c^+(k_1) > c^-(k_1) > \cdots > c^-(k_1,\cdots,k_{m-1}) > c^-(k_1,\cdots,k_m). \tag{10}$$

Suppose to the contrary that $c^+(k_1,\cdots,k_{j+1}) < c^+(k_1,\cdots,k_j)$. Then the cost (8) is reduced by postponing purchase of $s_+(k_1,\cdots,k_j)$ from time $t(k_1,\cdots,k_j)$ to the later time $t(k_1,\cdots,k_{j+1})$ (see (4)), so $s_+(k_1,\cdots,k_j)$ will be zero. Similarly, if $c^-(k_1,\cdots,k_{j+1}) > c^-(k_1,\cdots,k_j)$, the cost is lowered by postponing the sale of $s_-(k_1,\cdots,k_j)$ to the later time $t(k_1,\cdots,k_{j+1})$, so $s_-(k_1,\cdots,k_j)$ will be zero. Lastly, if $c_+(k_1,\cdots,k_j) < c_-(k_1,\cdots,k_j)$ one can make arbitrarily large profits simply by buying and selling equal amounts $s_+(k_1,\cdots,k_j) = s_-(k_1,\cdots,k_j)$, which is not meaningful. One consequence of (10) is that in any cost-minimizing policy, one will have

$$s_+(k_1,\cdots,k_j) = 0 \text{ or } s_-(k_1,\cdots,k_j) = 0. \tag{11}$$

Restriction on penalty The assumption is:

$$g(d,x) \text{ is nonnegative and convex in } x \text{ for each } d. \tag{12}$$

We give important forms of penalty below. In many cases $g(d,x)$ is decreasing in x.

Value of lost load (VOLL) With

$$g(d,x) = \gamma^+(k_1,\cdots,k_m)[d-x]_+, \tag{13}$$

$T_m[g(d(k_1,\cdots,k_m),\ x(k_1,\cdots,k_m))]$ is the value of lost load (VOLL) in $[k_1T_1+\cdots+k_mT_m,\ k_1T_1+\cdots+(k_m+1)T_m]$ when the supply is insufficient to serve the demand. $\gamma^+(k_1,\cdots,k_m)$ is the social cost of power interruption; its value is \$1,000-10,000 per MWh. g specified by (13) is decreasing and convex.

Loss of load probability (LOLP) Because it accounts for uncertainty, LOLP is considered a better measure of reliability than the traditional deterministic 'reserve margin', especially when intermittent sources are significant [5]. The use of LOLP to specify a risk target in system operations is also advocated in [12, 18]: the supply S must be such that the LOLP during the interval $[0,N_1T_1]$ is bounded by a pre-specified target value $\alpha \geq 0$. This is equivalent to the constraint

$$P\{d(k_1,\cdots,k_m) > x(k_1,\cdots,k_m) \mid Y(t(k_1,\cdots,k_m))\} \leq \alpha. \tag{14}$$

Define the loss function

$$g(d,x) = \begin{cases} \infty, & \text{if } P\{d(k_1,\cdots,k_m) = d > x(k_1,\cdots,k_m) = x \mid Y(t(k_1,\cdots,k_m))\} > \alpha \\ 0, & \text{otherwise} \end{cases}. \tag{15}$$

g specified by (15) is decreasing and convex in x. Resorting to infinite values in (15) can be eliminated by the following observation. The least supply x with LOLP at most α is the α-quantile Φ_α:

$$\Phi_\alpha(k_1,\cdots,k_m) = \inf\{x \mid P(d(k_1,\cdots,k_m) > x \mid Y(t(k_1,\cdots,k_m)) \leq \alpha\}. \tag{16}$$

$\Phi_\alpha(k_1,\cdots,k_m)$ is a function of $Y(t(k_1,\cdots,k_m))$, and the penalty (15) will be finite only if

$$x(k_1,\cdots,k_m) \geq \Phi_\alpha(k_1,\cdots,k_m), \quad \text{a.s.}. \tag{17}$$

Since $x(k_1,\cdots,k_m) = x(k_1,\cdots,k_{m-1}) + s(k_1,\cdots,k_m)$, the least-cost decision at stage $(k_1,\cdots,k_m)$ that satisfies (17) is

$$\begin{aligned} s^*_+(k_1,\cdots,k_m) &= [\Phi_\alpha(k_1,\cdots,k_m) - x(k_1,\cdots,k_{m-1})]_+, \\ s^*_-(k_1,\cdots,k_m) &= [\Phi_\alpha(k_1,\cdots,k_m) - x(k_1,\cdots,k_{m-1})]_-. \end{aligned}$$

The cost of this decision is $g(x(k_1,\cdots,k_{m-1}))$, wherein

$$g(x) = c^+(k_1,\cdots,k_m)[\Phi_\alpha(k_1,\cdots,k_m) - x]_+ + c^-(k_1,\cdots,k_m)[\Phi_\alpha(k_1,\cdots,k_m) - x]_-. \tag{18}$$

Hence we can replace the m-stage cost (8) with constraint (14) by the $9m-1$)-stage cost

$$J(\pi) = \mathbb{E}\left\{T_1 \sum_{k_1=0}^{N_1-1}[c^+(k_1)s_+(k_1) + c^-(k_1)s_-(k_1)] + \cdots + T_{m-1}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}\cdots\sum_{k_{m-1}=0}^{N_{m-1}-1}[g(x(k_1,\cdots,k_{m-1}))]\right\}.$$

Note that $g(x)$ given by (18) is convex and decreasing in x because, by (10), $c^-(k_1,\cdots,k_m) \le c^+(k_1,\cdots,k_m)$.

Reserve-at-risk (RaR) Inequality (17) is related to RaR introduced in [18] in analogy with 'value at risk' used in finance [17]. Call

$$r = r(k_1,\cdots,k_m) = x(k_1,\cdots,k_m) - d(k_1,\cdots,k_m)$$

the *reserve*; then $RaR_\alpha = x(k_1,\cdots,k_m) - \Phi_\alpha(k_1,\cdots,k_m)$ is the reserve-at-risk (at level α). At time $t(k_1,\cdots,k_m)$ the net demand $d(k_1,\cdots,k_m)$ and hence the reserve r may not be known. But because of (16), the reserve-at-risk RaR_α *is* known. As

$$P\{r < RaR_\alpha \mid Y(t(k_1,\cdots,k_m))\} = P\{d > \Phi_\alpha(k_1,\cdots,k_m) \mid Y(t(k_1,\cdots,k_m))\} \le \alpha,$$

the reserve r falls below RaR_α with probability at most α. Equivalently, (17) ensures that the reserve exceeds RaR_α with probability at least $(1-\alpha)$.

If there is agreement about the appropriate value of α (say, 1%), RaR_α can serve as a reserve target for the SO. At any time a negative RaR_α indicates that SO should increase the reserve by the same amount in order to maintain the required level of operational reliability, but with a positive RaR_α the operator knows that the reliability level is higher than required [18].

Conditional reserve at risk The event that the reserve drops below RaR_α occurs with probability α, and one measure of the risk when this event occurs is the conditional reserve at risk:

$$CRaR_\alpha = -E\{r \mid r < RaR_\alpha,\, Y(t(k_1,\cdots,k_m))\} = \frac{1}{\alpha}\int_{\Phi_\alpha}^{\infty}(\delta - x)p(\delta)d\delta, \tag{19}$$

in which $r = x - \delta = x(k_1,\cdots,k_m) - d(k_1,\cdots,k_m)$, $p(\delta) = p(d = \delta \mid Y(t(k_1,\cdots,k_m))$ is the probability density of the net demand d conditional on $Y(t(k_1,\cdots,k_m))$. The interpretation of (19) is this. Suppose at some time $t(k_1,\cdots,k_m)$, $RaR_\alpha = 100$MW, $CRaR_\alpha = 600$MW. Then the loss of load is expected to be 600MW if the reserve drops below 100MW. $CRaR_\alpha$ is decreasing and convex in x. There are instructive calculations of RaR_α and $CRaR_\alpha$ in [18].

Cost of excessive generation Random large amounts of wind power can cause net demand $d = d(k_1,\cdots,k_m)$ to drop below the supply $x = x(k_1,\cdots,k_m)$. A penalty may then be assessed

$$g(d,x) = \gamma^-(k_1,\cdots,k_m)[x-d]_+,$$

which may be combined with (13) into the convex penalty function

$$g(d,x) = \gamma^+[d-x]_+ + \gamma^-[x-d]_+.$$

In this case, it may be less costly to curtail wind at stage m, so RLD would select $s^*(k_1,\cdots,k_m) < 0$. Alternatively, one may replace (14) with

$$P\{d(k_1,\cdots,k_m) < x(k_1,\cdots,k_m) \mid Y(t(k_1,\cdots,k_m))\} \le \alpha,$$

and define the risk from wind on-ramps as a 'wind curtailment at risk' (WCaR) similarly to RaR.

2.4 Wind aggregator

A wind aggregator may contract to deliver blocks of firm power $L(k_1)$ at a fixed price $c^-(k_1)$ during $[k_1T_1,(k_1+1)T_1]$. The contracted power combines wind $W(t)$ and conventional power supply $S(t)$. The aggregator wants to minimize the expected net cost,

$$-\sum_{k_1} c^-(k_1)L(k_1) + J(\pi),$$

in which the first term is the negative of the revenue and the second term is the cost (8). The aggregator's RLD gives the optimum blocks $L^*(k_1)$ that the aggregator should contract. The aggregator's cost is well modeled by constant prices, whereas the SO's cost may be better specified by a cost function.

3 Optimality conditions

Most of this section is concerned with the case of constant prices. The case of a cost function is treated in Theorem 3. For any policy π the **cost-to-go** in state $x = x(k_1,\cdots,k_{j-1})$ at stage $(k_1,\cdots,k_j)$ conditional on the information available at $t(k_1,\cdots,k_j)$ is (see Figure 3)

$$\begin{aligned}
J(\pi,x,k_1,\cdots,k_j) &= \mathbb{E}\,\{T_j[c^+(k_1,\cdots,k_j)s_+(k_1,\cdots,k_j)+c^-(k_1,\cdots,k_j)s_-(k_1\cdots k_j)] \\
&+ T_{j+1}\sum_{k_{j+1}=0}^{N_{j+1}-1}[c^+(k_1,\cdots,k_{j+1})s_+(k_1,\cdots,k_{j+1})+c^-(k_1,\cdots,k_{j+1})s_-(k_1\cdots k_{j+1})] \\
&+ T_{j+2}\sum_{k_{j+1}=0}^{N_{j+1}-1}\sum_{k_{j+2}=0}^{N_{j+2}-1}[c^+(k_1,\cdots,k_{j+2})s_+(k_1,\cdots,k_{j+2})+c^-(k_1,\cdots,k_{j+2})s_-(k_1\cdots k_{j+2})] \\
&\cdots \\
&+ T_m\sum_{k_{j+1}=0}^{N_{j+1}-1}\cdots\sum_{k_m=0}^{N_m-1}[c^+(k_1,\cdots,k_m)s_+(k_1,\cdots,k_m)+c^-(k_1,\cdots,k_m)s_-(k_1,\cdots,k_m)] \\
&+ T_m\sum_{k_{j+1}=0}^{N_{j+1}-1}\cdots\sum_{k_m=0}^{N_m-1}[g(d(k_1,\cdots,k_m),\ x(k_1,\cdots,k_m))] \mid Y(t(k_1,\cdots,k_j))\}.
\end{aligned}$$

The **value function** in state $x = x(k_1,\cdots,k_{j-1})$ at stage $(k_1,\cdots,k_j)$ is defined as

$$J^*(x,k_1,\cdots,k_j) = \inf_{\pi} J(\pi,x,k_1\cdots k_j). \tag{20}$$

$J^*(x,k_1,\cdots,k_j)$ is a function of $Y(t(k_1,\cdots,k_j))$, that is, it is $\mathcal{Y}(t(k_1,\cdots,k_j))$-adapted.

Recalling the constraint (5), stochastic dynamic programming considerations [10] lead to the next result.

Lemma 1 *The value function satisfies the following backward iteration. In state $x = x(k_1,\cdots,k_{m-1})$ at the terminal stage $(k_1,\cdots,k_m)$,*

$$\begin{aligned}
J^*(x,k_1,\cdots,k_m) &= \inf_{s_+\geq 0\geq s_-}\{T_m[c^+(k_1,\cdots,k_m)s_+ + c^-(k_1,\cdots,k_m)s_-] \\
&+ T_m\mathbb{E}[g(d(k_1,\cdots,k_m),x+s_++s_-) \mid Y(t(k_1,\cdots,k_m))]\}\ a.s.
\end{aligned} \tag{21}$$

and for $j \le m-1$, in state $x = x(k_1,\cdots,k_{j-1})$ at stage $(k_1,\cdots,k_j)$,

$$\begin{aligned} J^*(x,k_1,\cdots,k_j) &= \inf_{s^+\ge 0\ge s^-} \big\{T_j[c^+(k_1,\cdots,k_j)s_+ + c^-(k_1,\cdots,k_j)s_-] \\ &+ \sum_{k_{j+1}=0}^{N_{j+1}-1} \mathbb{E}\big[J^*(x+s_++s_-,k_1,\cdots,k_{j+1}) \mid Y(t(k_1,\cdots,k_j))\big]\big\} \text{ a.s.} \end{aligned} \tag{22}$$

Moreover, the minimizing $s_+,\ s_-$ in (21)-(22) are the optimal decisions $s^*_+(k_1,\cdots,k_j),\ s^*_-(k_1,\cdots,k_j)$.

Definition For a convex, differentiable function $f(s),\ s \ge 0$, define

$$\nabla f(\varphi) = \begin{cases} \{(\partial f/\partial s)(\varphi)\} & \varphi > 0 \\ \{\gamma \le (\partial f/\partial s)(0)\} & \varphi = 0 \end{cases},$$

and its partial inverse $[\nabla f]^{-1}$:

$$[\nabla f]^{-1}(\gamma) = \min\{\varphi \ge 0 \mid \gamma \in \nabla f(\varphi)\}.$$

If $f(s)$ is random and adapted to the σ-field $\mathcal{Y}$ for each s, $[\nabla f]^{-1}(\gamma)$ is adapted to $\mathcal{Y}$ for each γ. If $\mathcal{Y}'$ is a sub-σ-field of $\mathcal{Y}$,

$$\nabla \mathbb{E}\{f \mid \mathcal{Y}'\}(\varphi) = \mathbb{E}\{\nabla f(\varphi) \mid \mathcal{Y}'\}.$$

Theorem 1 *The value function $J^*(x,k_1,\cdots,k_j),\ j \le m$, has the following properties.*
1. $J^*(x,k_1,\cdots,k_j)$ is $\mathcal{Y}(t(k_1,\cdots,k_j))$-adapted and convex.
2. There are $\mathcal{Y}(t(k_1,\cdots,k_j))$-adapted thresholds $\varphi_\pm(k_1,\cdots,k_j)$ that give the optimal decision in state $x = x(k_1,\cdots,k_{j-1})$ as

$$s^*_+(x,k_1,\cdots,k_j) = [\varphi_+(k_1,\cdots,k_j) - x]_+,\ \ s^*_-(x,k_1,\cdots,k_j) = [\varphi_-(k_1,\cdots,k_j) - x]_-. \tag{23}$$

3. The thresholds $\varphi_\pm(k_1,\cdots,k_j)$ do not depend on the state x. At the terminal stage

$$\varphi_\pm(k_1,\cdots,k_m) = \big[\nabla(T_m\hat{g}(d(k_1,\cdots,k_m)))\big]^{-1}(-T_m c^\pm(k_1,\cdots,k_m)), \tag{24}$$

$$\hat{g}(d(k_1,\cdots,k_m),z) = \mathbb{E}\{g(d(k_1,\cdots,k_m),z) \mid \mathcal{Y}(t(k_1,\cdots,k_m))\}, \tag{25}$$

and for $j \le m-1$,

$$\varphi_\pm(k_1,\cdots,k_j) = \big[\nabla(\sum_{k_{j+1}=0}^{N_{j+1}-1} \hat{J}(k_1,\cdots,k_{j+1}))\big]^{-1}(-T_j c^\pm(k_1,\cdots,k_j)) \tag{26}$$

$$\hat{J}(z,k_1,\cdots,k_{j+1}) = \mathbb{E}\{J^*(z,k_1,\cdots,k_{j+1}) \mid \mathcal{Y}(t(k_1,\cdots,k_j))\}. \tag{27}$$

4. The value function iteration for the terminal stage is

$$J^*(x,k_1,\cdots,k_m) = T_m[c^+(m)s^*_+(m) + c^-(m)s^*_-(m)] + T_m\mathbb{E}\{g(d(k_1,\cdots,k_m),x) \mid \mathcal{Y}(t(k_1,\cdots,k_m))\}, \tag{28}$$

and for $j \le m-1$ *it is*

$$\begin{aligned} J^*(x,k_1,\cdots,k_j) &= T_j[c^+(j)s_+^*(j)+c^-(j)s_-^*(j)] \\ &+ \sum_{k_{j+1}=0}^{N_{j+1}-1} \mathbb{E}\{J^*(x+s_+^*(j)+s_-^*(j),k_1,\cdots,k_{j+1}) \mid \mathcal{Y}(t(k_1,\cdots,k_j))\}, \end{aligned} \tag{29}$$

In (28)-(33) $c^{\pm}(j) = c^{\pm}(k_1,\cdots,k_j)$*,* $s_{\pm}^*(j) = s_{\pm}^*(x,k_1,\cdots,k_j),\ j \le m$*.*

5. Lastly, for all x and $\gamma \in \nabla J^*(x,k_1,\cdots,k_j)$

$$T_j c^-(k_1,\cdots,k_j) \le -\gamma \le T_j c^+(k_1,\cdots,k_j). \tag{30}$$

Proof See Appendix A. □

Theorem 2 gives a recursive 'closed-form' formula for the thresholds.

Theorem 2 *At the terminal stage,* $\varphi_{\pm}(m)$ *is the solution of the equation*

$$f_m(x) = T_m c^{\pm}(m), \tag{31}$$

with

$$f_m(x) = -T_m \mathbb{E}_m\{\nabla g(x)\}. \tag{32}$$

At stage $(k_1,\cdots,k_j),\ j \le m-1$*,* $\varphi_{\pm}(j)$ *is the solution of the equation*

$$f_j(x) = f(k_1,\cdots,k_j,x) = T_j c^{\pm}(j), \tag{33}$$

with

$$\begin{aligned}
& f_j(x) = \sum_{k_{j+1}=0}^{N_{j+1}-1} \Big[\\
& +T_{j+1}c^+(j+1)P_j(x \le \varphi_+(j+1)) \\
& +T_{j+1}c^-(j+1)P_j(x \ge \varphi_-(j+1)) \\
& + \sum_{k_{j+2}=0}^{N_{j+2}-1} \Big[T_{j+2}c^+(j+2)P_j(x \le \varphi_+(j+2), \varphi_+(j+1) < x < \varphi_-(j+1)) \\
& +T_{j+2}c^-(j+2)P_j(x \ge \varphi_-(j+2), \varphi_+(j+1) < x < \varphi_-(j+1)) \\
& + \sum_{k_{j+3}=0}^{N_{j+3}-1} \Big[T_{j+3}c^+(j+3)P_j(x \le \varphi_+(j+3), \varphi_+(j+2) < x < \varphi_-(j+2), \varphi_+(j+1) < x < \varphi_-(j+1)) \\
& +T_{j+3}c^-(j+3)P_j(x \ge \varphi_-(j+3), \varphi_+(j+2) < x < \varphi_-(j+2), \varphi_+(j+1) < x < \varphi_-(j+1)) \\
& \qquad \cdots \\
& + \sum_{k_m=0}^{N_m-1} \Big[T_m c^+(m)P_j(x \le \varphi_+(m), \varphi_+(m-1) < x < \varphi_-(m-1), \cdots, \varphi_+(j+1) < x < \varphi_-(j+1)) \\
& +T_m c^-(m)P_j(x \ge \varphi_-(m), \varphi_+(m-1) < x < \varphi_-(m-1), \cdots, \varphi_+(j+1) < x < \varphi_-(j+1)) \\
& -T_m \mathbb{E}_j\{\nabla g(x)\mathbf{1}(\varphi_+(m) < x < \varphi_-(m), \cdots, \varphi_+(j+1) < x < \varphi_-(j+1))\}\Big] \cdots \Big]\Big]
\end{aligned} \tag{34}$$

$$f_j(x) = -\mathbb{E}\{\sum_{k_{j+1}=0}^{N_{j+1}-1} \nabla J^*(x,k_1,\cdots,k_{j+1}) \mid \mathcal{Y}(t(k_1,\cdots,k_j))\}, \quad j \leq m-1. \tag{35}$$

The functions $f_j(x)$ are all non-negative, decreasing functions of x. Moreover

$$f_j(x) = -\mathbb{E}\{\sum_{k_{j+1}=0}^{N_{j+1}-1} \nabla J^*(x,k_1,\cdots,k_{j+1}) \mid \mathcal{Y}(t(k_1,\cdots,k_j))\}, \quad j \leq m-1. \tag{36}$$

Above

$$\nabla g(x) = \nabla g(d(k_1,\cdots,k_m),\ x),\ c^{\pm}(j) = c^{\pm}(k_1,\cdots,k_j),\ \varphi_{\pm}(j) = \varphi_{\pm}(k_1,\cdots,k_j);\ \textit{and}$$

$$E_j(\chi) = \mathbb{E}\{\chi \mid \mathcal{Y}(t(k_1,\cdots,k_j)\},\ \ P_j(A) = \mathbb{E}\{\mathbf{1}(A) \mid \mathcal{Y}(t(k_1,\cdots,k_j)\}$$

denote the conditional expectation and conditional probability of the random variable χ and event A.

Proof See Appendix B. □

Theorem 3 deals with the case when the marginal cost is not constant. The cost of policy π is

$$\begin{aligned} J(\pi) &= \mathbb{E}\,\Big\{T_1 \sum_{k_1=0}^{N_1-1} C(k_1;s_1(k_1)) + T_2 \sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} C(k_1,k_2;s(k_1,k_2)) \\ &+ \cdots \\ &+ T_m \sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}\cdots\sum_{k_m=0}^{N_m-1}[C(k_1,\cdots,k_m;s(k_1,\cdots,k_m)) + g(d(k_1,\cdots,k_m),\ x(k_1,\cdots,k_m))]\Big\}. \end{aligned}$$

Above, $C(k_1,\cdots,k_j;s)$ is convex in s, and $s > 0$ or < 0, accordingly as a block is purchased or sold. $g(d,\cdot)$ is the convex terminal penalty. The value function is defined in (20).

Lemma 1 holds with obvious changes: in state $x = x(k_1,\cdots,k_{m-1})$,

$$J^*(x,k_1,\cdots,k_m) = \inf_s\{T_m C(k_1,\cdots,k_m;s) + T_m \mathbb{E}[g(d(k_1,\cdots,k_m,x+s \mid Y(t(k_1,\cdots,k_m)]\}a.s. \tag{37}$$

and for $j \leq m-1$, in state $x(k_1,\cdots,k_{j-1})$ at stage $(k_1,\cdots,k_j)$,

$$J^*(x,k_1,\cdots,k_j) = \inf_s\{T_j C(k_1,\cdots,k_j;s) + \sum_{k_{j+1=0}}^{N_{j+1}-1} \mathbb{E}[J^*(x+s,k_1,\cdots,k_j) \mid Y(t(k_1,\cdots,k_j))]\}a.s. \tag{38}$$

Moreover, the minimizing s in (37)-(38) is the optimum decision.

Theorem 3 *The value function $J^*(x,k_1,\cdots,k_j)$, $j \leq m$, is convex and $\mathcal{Y}(k_1,\cdots,k_j)$-adapted. The optimum decision $s^*(k_1,\cdots,k_m)$ in state $x = x(k_1,\cdots,k_{m-1})$ at the terminal stage $(k_1,\cdots,k_m)$ is the solution s of*

$$\begin{aligned} -T_m c(k_1,\cdots,k_m;s) &= T_m \frac{\partial \hat{g}}{\partial s}(d(k_1,\cdots,k_m),x+s), \qquad (39) \\ \hat{g}(d,x+s) &= \mathbb{E}\{g(d,x+s) \mid \mathcal{Y}(t(k_1,\cdots,k_m))\}, \qquad (40) \end{aligned}$$

and for $j \leq m-1$, the optimal decision $s^*(k_1,\cdots,k_j)$ in state $x = x(k_1,\cdots,k_{j-1})$ at stage $(k_1,\cdots,k_j)$ is the solution s of

$$-T_j c(k_1,\cdots,k_j;s) = \frac{\partial}{\partial s}\sum_{k_{j+1}=0}^{N_{j+1}-1} \hat{J}(x+s,k_1,\cdots,k_{j+1}) \quad (41)$$

$$\hat{J}(x+s,k_1,\cdots,k_{j+1}) = \mathbb{E}\{J(x+s,k_1,\cdots,k_{j+1}) \mid \mathcal{Y}(t(k_1,\cdots,k_j))\} \quad (42)$$

Above $c(k_1,\cdots,k_j,s) = \partial C/\partial s(k_1,\cdots,k_j;s)$ denotes the marginal cost.

Proof See Appendix C. □

4 Gaussian forecast errors

Formulas (31)-(34) become very simple when forecast errors are Gaussian, as in [5, 2, 11, 6, 7]. The precise assumption is as follows.

Assumption
The net demand $d(k_1,\cdots,k_m)$ can be expressed as

$$d(k_1,\cdots,k_m) = \bar{d}(k_1,\cdots,k_m) + \varepsilon(k_1) + \cdots + \varepsilon(k_1,\cdots,k_m), \quad (43)$$

in which:

- $\bar{d}(k_1,\cdots,k_m)$ is $\mathcal{Y}(t(k_1))$ adapted;
- $\varepsilon(k_1,\cdots,k_j)$ is $\mathcal{Y}(t(k_1,\cdots,k_{j+1}))$ adapted and independent of $\mathcal{Y}(t(k_1,\cdots,k_j))$;
- $\varepsilon(k_1),\cdots,\varepsilon(k_1,\cdots,k_m)$ are independent Gaussian random variable with zero mean and standard deviation $\sigma(k_1,\cdots,k_j)$.

Under this assumption, the forecast of $d(k_1,\cdots,k_m)$ at time $t(k_1,\cdots,k_j)$, based on $\mathcal{Y}(t(k_1,\cdots,k_j))$, is

$$\mu(k_1,\cdots,k_j) = \bar{d}(k_1,\cdots,k_m) + \varepsilon(k_1) + \cdots + \varepsilon(k_1,\cdots,k_{j-1}). \quad (44)$$

The forecast error

$$d(k_1,\cdots,k_m) - \mu(k_1,\cdots,k_j) = \varepsilon(k_1,\cdots,k_j) + \cdots + \varepsilon(k_1,\cdots,k_m)$$

is Gaussian with zero mean and variance

$$\sigma^2(k_1,\cdots,k_j) + \cdots + \sigma^2(k_1,\cdots,k_m).$$

Thus the new observations at $t(k_1,\cdots,k_j)$ adds the 'correction' $\varepsilon(k_1,\cdots,k_{j-1})$ to the previous forecast $\mu(k_1,\cdots,k_{j-1})$ and reduces the error variance by $\sigma^2(k_1,\cdots,k_{j-1})$.

Lemma 2 *Suppose the penalty is (13):*

$$g(d(k_1,\cdots,k_m),x) = \gamma^+(k_1,\cdots,k_m)[d(k_1,\cdots,k_m) - x]_+.$$

Then there exist pre-computable deterministic functions $\psi(k_1,\cdots,k_j,\cdot)$ such that $f(k_1,\cdots,k_j,x)$ in (32), (33) is given by

$$f(k_1,\cdots,k_j,x) = \psi(k_1,\cdots,k_j,x-\mu(k_1,\cdots,k_j)). \quad (45)$$

Proof See Appendix D, which also shows how to calculate ψ. □

Theorem 4 *If the penalty is (13), the optimal thresholds are given by*

$$\phi_{\pm}(k_1,\cdots,k_j) = \mu(k_1,\cdots,k_j) + \Delta_{\pm}(k_1,\cdots,k_j), \tag{46}$$

in which the constant risk premiums $\Delta_{\pm}(k_1,\cdots,k_j)$ can be computed ahead of time.

Proof Substituting (45) into (31), (33) shows that $\phi_{\pm}(j)$ is the solution of

$$\psi(k_1,\cdots,k_j,x-\mu(k_1,\cdots,k_j)) = T_j c^{\pm}(k_1,\cdots,k_j)).$$

Hence by selecting $\Delta_{\pm}(k_1,\cdots,k_j)$ so that

$$\psi(k_1,\cdots,k_j,\Delta_{\pm}(k_1,\cdots,k_j))) = T_j c^{\pm}(k_1,\cdots,k_j)), \tag{47}$$

one obtains (46). □

The optimal policy is implemented by the following procedure.

Stage 0: Compute the risk premiums $\Delta_{\pm}(k_1,\cdots,k_j)$ by solving (47).

Stage k_1: At time $t(k_1)$, obtain the forecast $\mu(k_1)$, and calculate the thresholds

$$\phi_{\pm}(k_1) = \mu(k_1) + \Delta_{\pm}(k_1).$$

Suppose $z = z(k_1)$ is the initial supply. The optimal decision $s^*(z,k_1)$ is given by:

$$s^*(z,k_1) = \begin{cases} \phi_+(k_1) - z, & \text{if } \phi_+(k_1) - z > 0 \\ 0, & \text{if } \phi_+(k_1) < z < \phi_-(1) \\ -[z - \phi_-(k_1)], & \text{if } \phi_-(k_1) < z \end{cases} \tag{48}$$

Stage $k_1,\cdots,k_j$: At time $t(k_1,\cdots k_j)$ update the forecast $\mu(k_1,\cdots,k_j) = \mu(k_1,\cdots,k_{j-1}) + \varepsilon(k_1,\cdots,k_m)$, and calculate the thresholds

$$\phi_{\pm}(k_1,\cdots,k_j) = \mu(k_1,\cdots,k_j) + \Delta_{\pm}(k_1,\cdots,k_j).$$

The optimal decision in state $x = x(k_1,\cdots,k_{j-1})$ is given by:

$$s^*(x,k_1,\cdots,k_j) = \begin{cases} \phi_+(k_1,\cdots,k_j) - x, & \text{if } \phi_+(k_1,\cdots,k_j) - x > 0 \\ 0, & \text{if } \phi_+(k_1,\cdots,k_j) < x < \phi_-(k_1,\cdots,k_j) \\ -[x(k_1,\cdots,k_{j-1}) - \phi_-(k_1,\cdots,k_j)], & \text{if } \phi_-(k_1,\cdots,k_j) < x \end{cases} \tag{49}$$

If the supply x acquired up to stage $(k_1,\cdots,k_j)$ is smaller than the lower threshold, an energy block equal to the shortfall must be purchased; if the acquired supply is larger than the upper threshold, the excess must be sold.

5 Two examples

Two examples are worked out in this section. The first uses formulas (32)-(34). The second uses formula (46) for the Gaussian case. To simplify the notation and calculations we assume

$$T_1 = \cdots = T_m = 1;\ N_1 = \cdots = N_m = 1;\ c_-(1) = \cdots = c_-(m) = 0.$$

We write $c_+(k) = c(k)$. Because $c_-(j) = 0$, it does not pay to sell any capacity, so $s^*_-(j) = 0$, and we write $s(j) = s_+(j),\ s_-(j) = 0$. The decision $s(j)$ is taken at time $t(j)$. The thresholds are denoted $\varphi_+(j) = \varphi(j)$ and $\varphi_-(j) = \infty$ (which is equivalent to $s^*_-(j) = 0$).

The cost of policy π is

$$J(\pi) = \mathbb{E}\{\sum_{j=1}^{M} c(j)s(j) + g(d(m),x(m))\}.$$

The RLD dispatch is

$$s^*(j) = [\varphi(j) - x(j-1)]_+.$$

We assume that $d(m)$ is observed at the last decision time $t(m)$, and require the net demand to be met with probability one. So the penalty is (15) with $\alpha = 0$. Write $c(m+1) = \infty$. Then $\nabla g(d,x) = -c(m+1)\mathbf{1}(x < d)$, so (31)-(32) give the threshold in stage m,

$$\varphi(m) = d(m). \tag{50}$$

Henceforth we write $d(m) = d$. For $j \le m-1$, (31)-(34) show that the threshold $\varphi(j)$ is the solution x of the equation

$$\begin{aligned} c(j) &= c(j+1)P_j(x \le \varphi(j+1)) + c(j+2)P_j(x \le \varphi(j+2), \varphi(j+1) < x) + \cdots \\ &+ c(m)P_j(x \le \varphi(m), \varphi(m-1) < x, \cdots, \varphi(j+1) < x) \\ &+ c(m+1)P_j(x < d, \varphi(m) < x, \cdots, \varphi(j+1) < x). \end{aligned} \tag{51}$$

Divide both sides by $c(j+1)$; then $\varphi(j)$ is the solution of the equation

$$F_j(x) = \frac{c(j)}{c(j+1)}, \tag{52}$$

in which

$$\begin{aligned} F_j(x) &= P_j(x \le \varphi(j+1)) + \frac{c(j+2)}{c(j+1)}P_j(x \le \varphi(j+2), \varphi(j+1) < x) + \cdots \\ &+ \frac{c(m)}{c(j+1)}P_j(x \le \varphi(m), \varphi(m-1) < x, \cdots, \varphi(j+1) < x) \\ &+ \frac{c(m+1)}{c(j+1)}P_j(x < d, \varphi(m) < x, \cdots, \varphi(j+1) < x). \end{aligned} \tag{53}$$

In (53), $P_j(A)$ is the probability of event A, conditioned on $Y(t(j))$. Expression (53) is evaluated in reverse order $j = m-1, \cdots, 1$, starting with $\varphi(m) = d$. At stage $(m-1)$

$$F_{m-1}(x) = P_{m-1}(x \le \varphi(m)) = P\{x \le d \mid Y(t(m-1))\} = \frac{c(m-1)}{c(m)}. \tag{54}$$

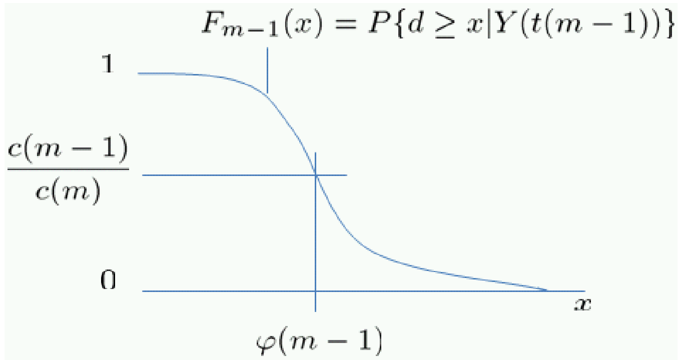

Figure 4: Threshold $\varphi(m-1)$ is the $c(m-1)/c(m)$-quantile of F_{m-1}.

Figure 4 depicts $F_{m-1}(x)$, and shows that $\varphi(m-1)$ is the $c(m-1)/c(m)$-quantile of $F_{m-1}(x) = P\{d \geq x \mid Y(t(m-1)\}$. $\varphi(m-1)$ is a function of $Y(t(m-1))$.

We now evaluate $F_{m-2}(x)$ from (53):

$$\begin{aligned} F_{m-2} &= P_{m-2}(x \leq \varphi(m-1)) + \frac{c(m)}{c(m-1)} P_{m-2}(x \leq \varphi(m), \varphi(m-1) < x) \\ &= P\{x \leq \varphi(m-1) \mid Y(t(m-2))\} + \frac{c(m)}{c(m-1)} P\{\varphi(m-1) < x < d \mid Y(t(m-2))\} \\ &= \frac{c(m-2)}{c(m-1)}. \end{aligned} \tag{55}$$

Thus $\varphi(m-2)$ is the $c(m-2)/c(m-1)$-quantile of the complementary distribution function $F_{m-2}(x)$. From (55) we see that to calculate $F_{m-2}(x)$ one needs the joint distribution of $\varphi(m-1)$ and $\varphi(m) = d$, conditioned on $Y(t(m-2))$. In general, the calculation of $\varphi(j)$, $F_j(x)$ needs the joint distribution of $\varphi(m) = d, \varphi(m-1), \cdots, \varphi(j+1)$, conditioned on $Y(t(j))$. The calculation can be automated to fit the form of the observations.

5.1 Example 1

Take $m = 3$, $c(1) = 50, c(2) = 100, c(3) = 1,000$. Purchases are made at $t(1), t(2), t(3)$. At $t(3)$, d is observed, so

$$\varphi(3 \mid d) = d.$$

At $t(2)$ a weather forecast arrives. It has two possible values, L (low demand) or H (high demand), each with probability 0.5. The distribution of d conditioned on the forecast is

$$P(d \mid L) = U[-2,1],\ P(d \mid H) = U[-1,2].$$

($U[a,b]$ is the uniform distribution over $[a,b]$.) The threshold at stage 2 is given by solving (55):

$$F_2(x \mid \omega) = P\{x \leq d \mid \omega\} = c(2)/c(3) = 100/1000 = 0.1,\ \omega = L, H. \tag{56}$$

Using $P(d \mid L) = U[-2,1]$, $P(d \mid H) = U[-1,2]$ in (55) gives an explicit expression for $F_2(x \mid \omega)$. The result if plotted in the right panel of Figure 5. Solving for the 0.1-quantiles in (56) gives the stage 2 threshold:

$$\varphi(2 \mid L) = 0.7,\ \varphi(2 \mid H) = 1.7.$$

We proceed to stage 1 assuming no information is available ($Y(t(1)) = \emptyset$), so (54) is

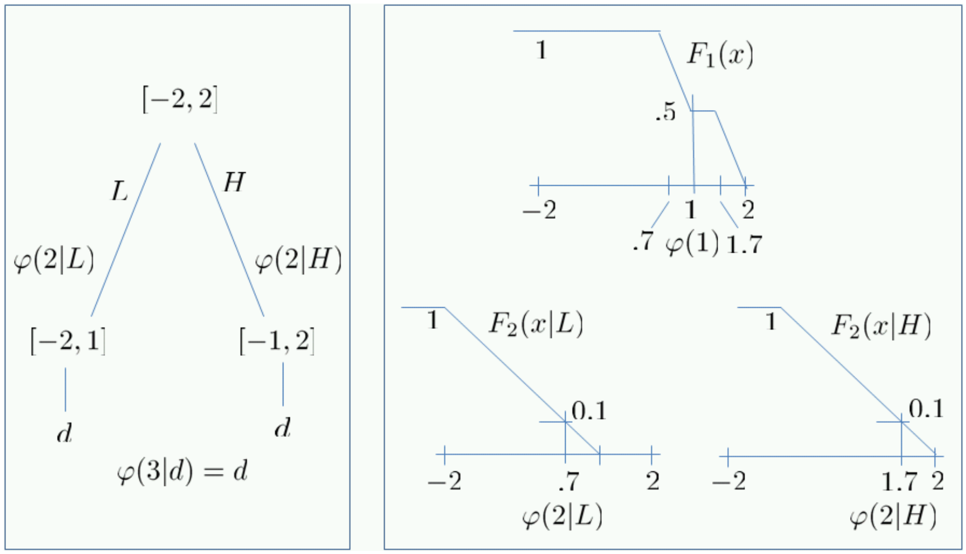

Figure 5: Information structure (left) and thresholds (right) for Example 1.

$$
\begin{aligned}
F_1(x) &= P(x \le \varphi(2)) + \frac{c(3)}{c(2)} P(\varphi(2) < x < d) \\
&= 0.5[P(x \le \varphi(2) \;\; L) + P(x \le ff(2) \mid H)] \\
&+ \; 0.1 \times 0.5[P(\varphi(2) < x < d \mid L) + P(\varphi(2) < x < d \mid H)] = \frac{c(1)}{c(2)} = 0.5.
\end{aligned}
$$

Substituting for the probabilities and the threshold $\varphi(2)$ gives the expression for $F_1(x)$, which is plotted in the right panel. By (54), the optimal threshold $\varphi(1)$ is the $c(1)/c(2) = 0.5$-quantile of $F_1(x)$, which can be read off from Figure 5: $\varphi(1) = 1$.

Thus for this example the RLD dispatch is:
Stage 1 Purchase $s^*(1) = \varphi(1) = 1$. So $x(1) = 1$.
Stage 2 Obtain forecast ω:
If $\omega = L, \varphi(2 \mid L) = 0.7$. Purchase $s^*(2) = [0.7 - x(1)]_+ = 0$.
If $\omega = H, \varphi(2 \mid H) = 1.7$. Purchase $s^*(2) = [1.7 - x(1)]_+ = 0.7$.
Stage 3 Observe d. Purchase $s^*(3) = [d - (x(1) + s^*(2))]_+$.

Suppose now that the weather forecast or stage 2 is not available. From (54), the optimal threshold $\varphi(1)$ is the solution of

$$\frac{c(1)}{c(3)} = 0.5 = P(x \le d). \tag{57}$$

Here $P(d)$ is is the prior distribution of d, so $P(d) \approx 0.5 \times U[-2,1] + 0.5 \times U[-1,2]$. Substituting this in (57) gives

$$\varphi(1) = 1.67.$$

Stage 1 Purchase $s^*(1) = \varphi(1) = 1.67$.
Stage 3 Observe d. Purchase $s^*(3) = [d - \varphi(1)]_+ = [d - 1.67]_+$.

The minimum expected cost is smaller when the stage 2 forecast is available. Surprisingly, the cost savings occur when net demand is low. For instance if $d < 1$ (which occurs with probability 5/6), the the after-forecast decision $s^*(1) = 1$ is better than the wasteful no-forecast decision $s^*(1) = 1.67$. But for $d > 1.67$, (which occurs with probability 1/18), the no-forecast decision is better, since it purchases 1.67 units at a lower cost. Observe that the currently practiced decoupled dispatch ignores the existence of the later decision, so its decision in stage 1 would be 1.67.

5.2 Example 2

We now consider several examples in which the forecast errors are Gaussian, although the net demand $d(m) = d$ need *not* be Gaussian. Theorem 4 applies, and the thresholds are given by (46):

$$\varphi(j) = \mu(j) + \Delta(j). \tag{58}$$

The forecast $\mu(j)$, based on $Y(t(j))$, has the error

$$d - \mu(j) = \varepsilon(j) + \cdots + \varepsilon(m); \tag{59}$$

$\varepsilon(j), \cdots, \varepsilon(m)$ are independent zero-mean Gaussian random variables with variances $\sigma^2(j), \cdots, \sigma^2(m)$.

At stage m, d is observed, so $\mu(m) = d$, $\sigma^2(m) = 0$, $\varphi(m) = d$. At stage $(m-1)$ by (54),

$$F_{m-1}(x) = P\{d \leq x \mid Y(t(m-1))\} = \frac{c(m-1)}{c(m)}, \tag{60}$$

and from (59)

$$P(d - \mu(m-1) \mid Y(t(m-1))\} \approx \mathcal{N}(0, \sigma^2(m-1)),$$

in which $\mathcal{N}$ is the Gaussian distribution. So $F_{m-1}(x)$ is given explicitly as

$$\begin{aligned} F_{m-1}(x) &= \frac{1}{\sqrt{2\pi\sigma^2(m-1)}} \int_{x-\mu(m-1)}^{\infty} exp - \frac{t^2}{2\sigma^2(m-1)} dt \\ &= 1 - \Phi(\frac{x - \mu(m-1)}{\sigma(m-1)\sqrt{2}}. \end{aligned} \tag{61}$$

Here Φ is the error function. From (46) the threshold at stage $(m-1)$ is

$$\varphi(m-1) = \mu(m-1) + \Delta(m-1), \tag{62}$$

and from (60) $\Delta(m-1)$ is the solution of

$$\frac{c(m-1)}{c(m)} = 1 - \Phi(\frac{\Delta(m-1)}{\sigma(m-1)\sqrt{2}}). \tag{63}$$

Thus $\Delta(m-1)$ can be computed ahead of time, as expected from Theorem 4.

Proceeding to stage $m-2$, from (55)

$$F_{m-2} = P_{m-2}(x \leq \varphi(m-1)) + \frac{c(m)}{c(m-1)} P_{m-2}(x \leq \varphi(m), \varphi(m-1) < x). \tag{64}$$

We evaluate the two probabilities in (64) using (62) and (59):

$$\begin{aligned} P_{m-2}(x \le \varphi(m-1)) &= P\{x-\mu(m-2) \le \varepsilon(m-2)+\Delta(m-1) \mid Y(t(m-2))\}, \\ P_{m-2}(\varphi(m-1) < x < \varphi(m)) &= P\{\varepsilon(m-2)+\Delta(m-1) < x-\mu(m-2) < \varepsilon(m-1)+\varepsilon(m-2) \mid Y(t(m-2))\}. \end{aligned}$$

Since conditional on $Y(t(m-2))$, $\varepsilon(m-1), \varepsilon(m-2)$ are independent Gaussian random variables with zero mean and variances $\sigma^2(m-1), \sigma^2(m-2)$, we can evaluate these two probabilities, and then express $F_{m-2}(x)$ as a function $\Psi_{m-2}(x-\mu(m-2))$ that does not depend on $Y(t(m-2))$. $\Delta(m-2)$ is then obtained by solving

$$\Psi_{m-2}(\Delta(m-2)) = \frac{c(m-2)}{c(m-1)}. \tag{65}$$

The optimum threshold is then given by

$$\varphi(m-2) = \mu(m-2) + \Delta(m-2).$$

As expected, the risk premium $\Delta(m-2)$ can be pre-computed from (65). We can proceed in this way through stages $m-3, \cdots, 1$. We consider some numerical examples.

Example 2A: 2-stage vs 10-stage dispatch We first specify the different components of the model. As before, the net demand $d(m) = d$ must be met with probability 1. d can take positive or negative values. We divide d by its maximum value d_{max}, so $-1 \le d \le 1$, but nevertheless refer to d/d_{max} as d.

For the remainder of this section $t(j)$ denotes the time horizon (previously $t - t(j)$) for stage j, so $t(j) \to 0$ as $j \to m$.

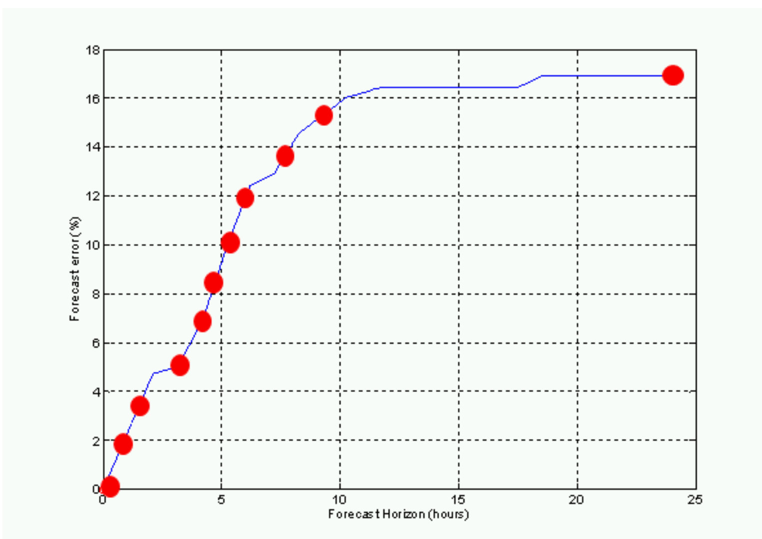

Figure 6: Forecast error vs forecast horizon (hours). Source: Iberdrola Renewables

Forecast error The percentage error as a function of forecast horizon from 0 to 24 hours (day-ahead) is given by the plot of Figure 6. The errors are Gaussian, and the percentage error is the ratio of the standard deviation to d_{max}. Since d is normalized, the percentage error is also the standard deviation of the forecast error. For any horizon t, $\mu(t)$ is the forecast, $\varepsilon(t)$ is its error, and $\sigma(t)$ is its standard deviation. That is,

$$\mu(t) = d - \varepsilon(t), \; \mathbb{E}\varepsilon(t) = 0, \mathbb{E}\varepsilon^2(t) = \sigma^2(t).$$

For example, from the plot, $\sigma(24) = 0.17, \sigma(9.1667) = 0.161$, and so on. These forecast errors are considered the best currently possible. By contrast, CAISO regards any forecast with a horizon of 6 hours as unreliable, whereas according to the figure the error at six hours is only $\sigma(6) = 0.09$.

Stages We consider a maximum of 10 stages. The time of the stages are chosen so that each stage reduces the uncertainty by 10%. That is, the time horizons are obtained by dividing the maximum uncertainty in the plot (which is 17% 24-hours ahead) by 10 and taking the corresponding 'forecast horizon' intercepts. This procedure leads to the following sequence of horizons:

$$24.00; 9.17; 7.62; 5.97; 5.30; 4.72; 4.10; 3.20; 1.52; 0.75; 0.0003.$$

Thus, the maximum percentage error 24-hours ahead is 17%, percentage error in stage 1 (9.1667- hours ahead) has decreased to $(0.9 \times 17)\%$, and so on. At stage 10 (0.0003-hours ahead), the error is 0% i.e. the net demand is completely known. The stages are shown by the circles in the plot.

Several optimal RLD strategies are considered. They differ in the subset of the 10 stages for which intra-day markets are available. As a benchmark we also consider the *oracle* case in which demand is known 24 hours ahead.

Prices The forward prices for the intra-day markets are determined as follows. From CAISO data, we take the 24-hours ahead purchase price to be \$52 per MWh, the load-following price (5-min ahead) to be \$60 per MWh and the real-time price (at stage 10) to be \$72 per MWh, corresponding to a load-following and regulation-up reserve price of \$8 and \$20 per MW, respectively. The price at any stage with time horizon t hours is obtained by an exponential fit: $p(t) = A + Be^{-\gamma t}$. That is, A, B, γ are selected so that $p(24) = 52, p(5 \text{ min}) = 60, p(0) = 72$. CAISO prices are averages over the year: on any given day they deviate considerably from these averages. This is useful to note since the optimal dispatch depends significantly on these prices. Suppose $s(j)$ is purchased in stage j. We require $\sum_1^{10} s(j) - d \geq 0$ wp 1. In Example 2B the zero LOLP constraint is replaced by a VOLL penalty.

RLD strategies The optimal dispatch is calculated and compared for three cases:
1. 2-stages: stages 1 (24-ahead) and 10 (real-time).
2. 10-stages: all stages in the plot.
3. Oracle: at stage 1, d is known, so oracle strategy purchases $s(1) = d_+$, at the lowest price.

The minimum cost for strategies 1,2 and 3 is successively lower. The RLD decision at stage j is

$$s^*(j) = [\varphi(j) - x(j-1)]_+ = \mu(j) + \Delta(j). \tag{66}$$

The risk premium $\Delta(j)$ is pre-calculated.

Results We make no assumptions about the distribution of d. Instead we calculate the optimal cost conditional on the realization of d as follows.
1. Select any realization of $d \in [-1, 1]$.
2. Generate 100 independent samples of the errors $\varepsilon(j)$ according to $\mathcal{N}(0, \sigma^2(j))$.
3. For each d and error sample calculate the corresponding realizations of the forecast $\mu(j)$ from (59), the optimal decision from (66), and the cost of the sample realization,

$$c(1)s^*(1) + \cdots + c(m)s^*(m).$$

The average of these 100 random costs is an estimate of the minimum expected cost for each RLD strategy, conditioned on d. Figure 7 displays their costs.

In Figure 7 the cost is normalized by \$72. Since 52/72 = 0.72, and the oracle strategy purchases d_+, its cost is $0.72 \times d_+$, which is the bottom or red plot. The green and blue plots are the 2-stage and 10-stage costs,

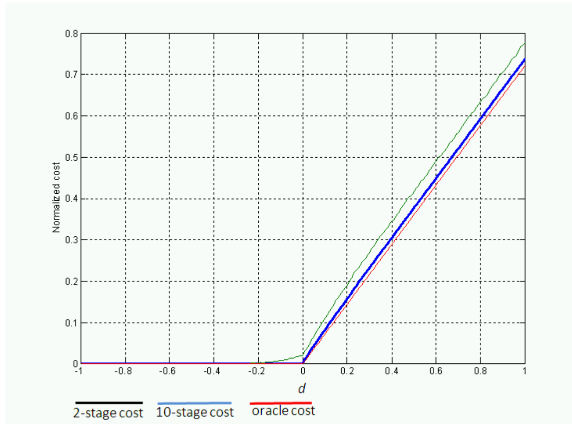

Figure 7: Minimum expected normalized cost conditioned on d, for three strategies. The unnormalized cost is obtained by multiplying by \$72.

respectively. As expected, the blue curve is below the green curve. Both strategies make purchases even if (in hindsight) d turns out to be negative. (Unlike the red plot, the green and red plots are not straight lines, despite their appearance.) The normalized 10-stage cost is lower than the 10-stage cost by about 0.05 or $(0.05 \times 72) = 3.6$ \$/MWh. This is a large savings in view of estimates of operating energy reserve costs of 2.5-5.0 \$/MWh [15].

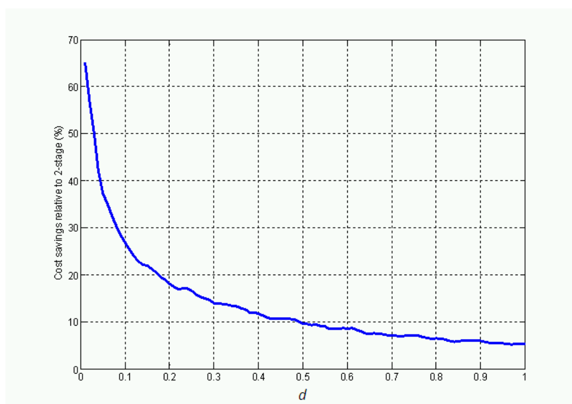

Figure 8: Relative savings of 10-stage vs. 2-stage strategy, conditioned on d.

Figure 8 is a plot of the relative savings, defined as the ratio (2-stage cost - 10-stage cost)/(2-stage cost), conditioned on d. The savings approach 70% for small values of d, and decrease for d positive and large, as more and more purchases in the 10-stage problem are made towards real time. Thus the 2-stage is relatively more wasteful (excessive reserves are purchased) when d is small. At $d = 0.4$ the difference is about 0.5, that is, the 10-stage dispatch purchases only one-half the energy of the 2-stage dispatch.

Figure 9 plots the additional minimum expected cost of the 2-stage and 10-stage strategies relative to the oracle strategy. Again we see that the relative impact of additional information and more stages is more pronounced for small values of d.

The impact of more stages and additional information can also be assessed from Figure 10, which plots the risk premium $\Delta(i)$ vs stage number. (Stage 11 is for $t = 0$.) The premium is large and negative in the early stages, approaching 0 at real time. The negative premium means (see (66)) that it is optimal to maintain

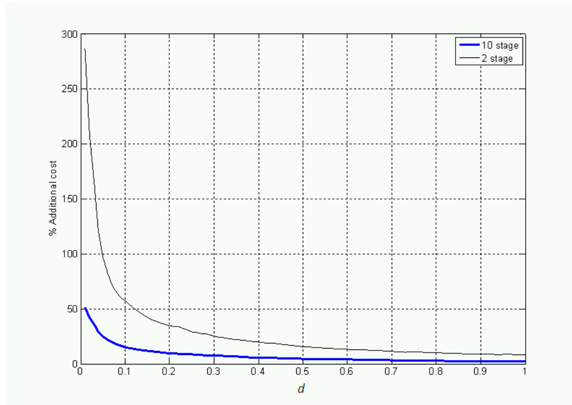

Figure 9: Additional cost of 2-stage and 10-stage strategies relative to oracle strategy.

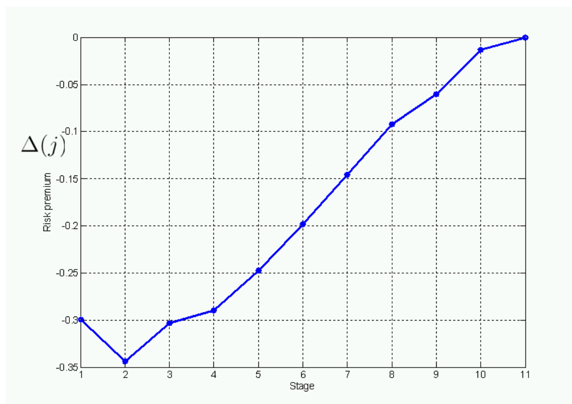

Figure 10: Risk premium vs stage number.

reserves *below* the forecast demand since the forecast may turn out to be too high, with compensating purchases at later stages.

Example 2B: 2-stage vs 4-stage and VOLL In Example 2A the last stage (stage 10) is at real time, when there is no forecast error and the reserve cost is \$72 per MWh. In practical terms, this should be taken as the cost of lost load, valued at \$1,000-10,000 per MWh. We compare three RLD dispatches. In the 2-stage problem, the stage 1 cost is \$52/MWh, and VOLL is \$1,000/MWh. The 4-stage problem has two additional stages: a 1-hour ahead stage at a cost of \$60/MWh and a 15-min ahead stage at a cost of \$72/MWh. In the oracle case, the 24-ahead forecast has no error. Figure 11 provides a comparison of the minimum expected cost, conditioned on d, for the three optimal dispatches. The actual cost is obtained by multiplying the normalized cost by \$1,000.

The effect of a VOLL penalty is gauged by comparing the plots of Figure 11 with those in Figure 7. Evidently, a large VOLL increases the value of additional information and more stages. The additional normalized cost incurred by the 2-stage dispatch is at 0.015 or \$15 per MWh. (This is much larger than the range 2.5.5.0 \$/MWh, which does not include VOLL.) Again, the 2-stage cost is higher when net demand is low. For example, when $d = 0$, the minimum cost for the 2-stage dispatch is \$15 (0.015×1000) vs \$1 for the 4-stage dispatch.

Example 2C: 2-stage vs 3-stage, different forecast errors In the last example we consider a 2-stage

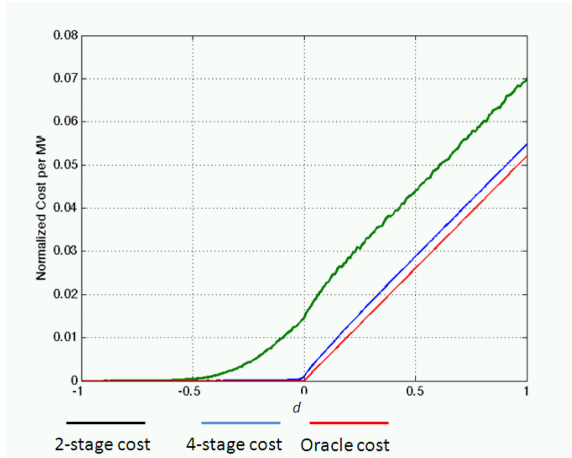

Figure 11: Minimum expected normalized cost conditioned on d, for three strategies. The \$ cost is obtained by multiplying by \$1,000/MWh.

(stages 1,2) vs a 3-stage (stages 1,2,3) RLD dispatch with costs of \$52, \$60 and \$72, and a VOLL of \$1000. We consider three levels of forecast error: best case denoted 1X (standard errors of Figure 6), medium case denoted 2X with twice the standard error, and the worst case denoted 3X with thrice the standard error.

Figure 12 gives the additional cost in the three cases incurred by the 2-stage dispatch, conditioned on the net demand d. (Note: the vertical scales are different.) The benefits are large: the 2-stage cost is larger than the 3-stage cost by about \$2.5 per MWh for the 1X case, \$4.5 per MWh for the 2X case and \$9.0 per MWh for the 3X case. This shows that having more stages (closer to real time) compensates for better forecasts.

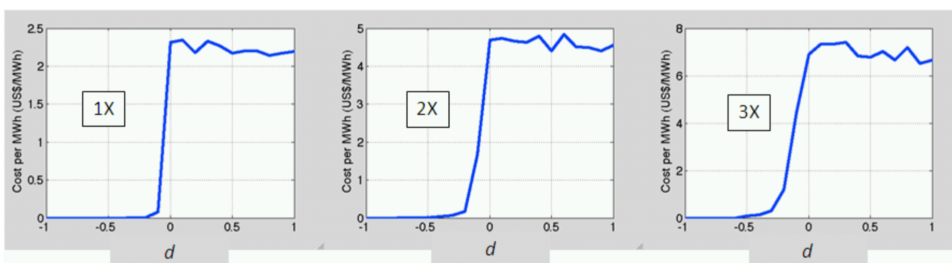

Figure 12: Comparison of minimum average cost for 2-stage vs 3-stage dispatch, conditioned on d for best (1X), medium (2X), and worst (3X) forecast errors.

6 Conclusions

A system operator's dispatch is a sequence of decisions to balance the supply and load of electric power at every instant, in the face of forecast errors about future load and renewable generation. The decisions comprise purchases of blocks of energy and reserve capacity (call options) in different markets, with each block getting shorter as real time approaches. The decisions are based on the information available at the times when the relevant market closes. In current practice, the decisions are decoupled: the SO minimizes the expected cost of each decision, ignoring the fact that subsequent decisions can correct for current errors. The present paper formulates the dispatch as a stochastic control problem, which takes future decisions into account in determining the current decision.

The optimal or risk-limiting dispatch (RLD) is found in 'closed-form' as a solution of the corresponding stochastic dynamic program. In the case of constant prices, RLD has a very attractive form: at each stage, one calculates two thresholds based on the available information. The SO's task is to acquire or sell energy so that the accumulated energy is between the two thresholds. In the case that the forecast errors are Gaussian, the thresholds can be pre-computed. When the SO faces a cost function, the RLD calculations are more complicated.

The closed-form formulas permit the comparison of optimal dispatch processes that in terms of the number of stages, price and penalty structure and the available information. As expected, the cost is lower as the number of stages increase or the forecast errors decrease. The cost reductions can be large: the addition of one or two intra-day markets (with correspondingly better forecasts) can reduce reserve energy costs by 50 percent in comparison with published estimates.

The comparison suggests the strong benefits that the California Independent System Operator or CAISO can gain by adding intra-day and intra-hour decisions to its current day-ahead and real-time balancing markets. Adding these decisions means that CAISO's traditional separation of 24 hour-ahead, 5-min load-following, and regulation power (which is reflected in a corresponding separation of generation resources) may no longer be appropriate. Instead, the SO should simply group those resources by how quickly they can supply power. The benefits of better forecasts is equally significant. The costs of these innovations remain unexplored.

In addition to system operators, the models and results in the paper may be used to formulate the problem faced by a wind 'aggregator' who must absorb the wind power variability with reserve energy, and sell the resulting firm power.

The paper has at least three limitations. Unit commitment or generator start-up costs are not included, because they add integer-valued decision variables. However, the results presented here remain useful after the unit commitment decisions. Transmission constraints are ignored in the paper's single-bus model. Lastly, ramping constraints are also not treated. Efforts to overcome these limitations are ongoing.

Appendix

A Proof of Theorem 1

The proof is based on Lemma 3.

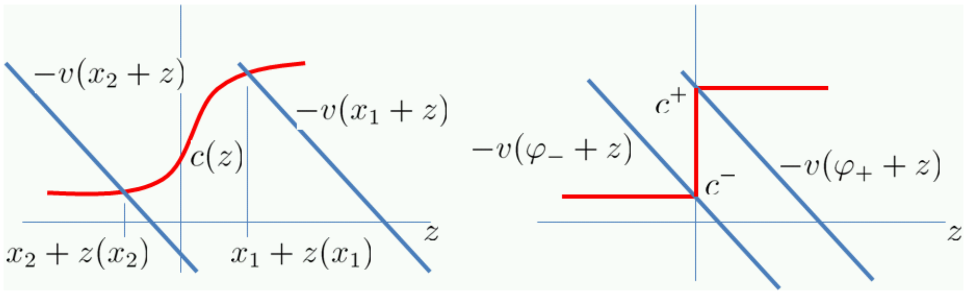

Figure 13: Lemma 3 (left). Corollary 1 (right):Threshold rule for constant prices.

Lemma 3 *Let $V(x)$, $x \geq 0$, be $\mathcal{Y}$-adapted and convex. Let $C(z)$, $z \in R$, be convex. Let $\mathcal{Y}' \subset \mathcal{Y}$ be a sub-σ-field. Denote*

$$c(z) = \frac{\partial C}{\partial z}(z),\ \hat{V}(x) = \mathbb{E}\{V(x) \mid \mathcal{Y}'\},\ v(x+z) = \frac{\partial \hat{V}}{\partial z}(x+z).$$

Then

$$z(x) = \arg\min_z \mathbb{E}\{C(z) + V(x+z) \mid \mathcal{Y}'\} \tag{67}$$

is the solution of

$$c(z(x)) + v(x+z(x)) = 0. \tag{68}$$

Moreover,

$$-1 \leq \frac{\partial z}{\partial x}(x) \leq 0. \tag{69}$$

Lastly

$$W(x) = \min_z \mathbb{E}\{C(z) + V(x+z) \mid \mathcal{Y}'\} = C(z(x)) + \hat{V}(x+z(x)) \tag{70}$$

is convex and

$$\frac{\partial W}{\partial x}(x) = v(x+z(x)) = -c(z(x)). \tag{71}$$

Proof Since $z \mapsto \mathbb{E}\{C(z) + V(x+z) \mid \mathcal{Y}'\}$ is convex, $z(x)$ is given by the first-order condition (68). Since C, $\hat{V}$ are convex, $z \mapsto c(z)$ is increasing and $z \mapsto -v(x+z)$ is decreasing in x. The intersection of these two curves occurs at $z(x)$ (see Figure 13, left). Differentiating (68) with respect to x gives

$$\frac{\partial z}{\partial x}(x) = -\frac{[\partial v/\partial x](x+z(x))}{[\partial c/\partial z](z(x)) + [\partial v/\partial x](x+z(x))}, \tag{72}$$

which implies (69) since $\partial c/\partial z$, $\partial v/\partial x$ are positive. Lastly, from (68) and (71)

$$\frac{\partial W}{\partial x}(x) = c(z(x))\frac{\partial z}{\partial x} + v(x+z(x))[1+\frac{\partial z}{\partial x}] = v(x+z(x)) = -c(z(x)),$$

which is increasing in x. Hence $W(x)$ is convex in x. □

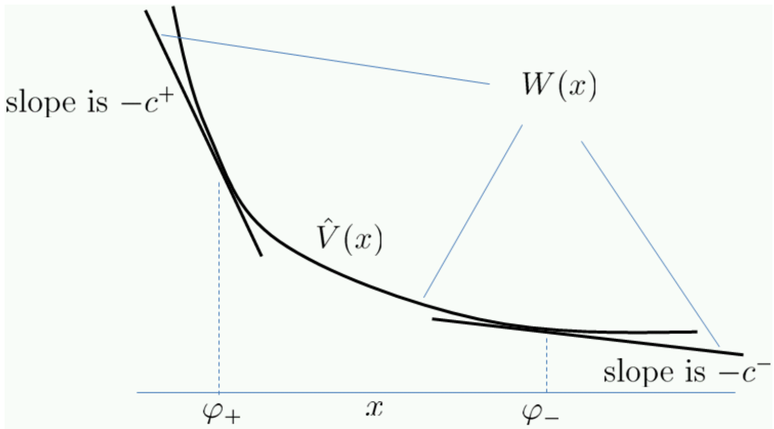

Figure 14: Proof of Corollary 1.

Corollary 1 *Suppose $C(z) = c^{+}z_{+} + c^{-}z_{-}$ (with $c^{+} \geq c^{-} \geq 0$). Define thresholds φ_{+}, φ_{-} by*

$$\frac{\partial \hat{V}}{\partial x}(\varphi_{+}) = -c^{+},\ \frac{\partial \hat{V}}{\partial x}(\varphi_{-}) = -c^{-}\ \textit{or}\ \varphi_{+} = [\nabla\hat{V}]^{-1}(-c^{+}),\ \varphi_{-} = [\nabla\hat{V}]^{-1}(-c^{-}).$$

Then

$$\begin{aligned} z(x)_{+} &= [\varphi_{+} - x]_{+},\ z(x)_{-} = [\varphi_{-} - x]_{-}, \\ \frac{\partial W}{\partial x}(x) &= -c^{+}\mathbf{1}(x < \varphi_{+}) + \frac{\partial \hat{V}}{\partial x}(x)\mathbf{1}(\varphi_{+} \leq x \leq \varphi_{-}) - c^{-}\mathbf{1}(\varphi_{-} < x), \\ W(x) &= c^{+}[\varphi_{+} - x]_{+} + \hat{V}(x + [\varphi_{+} - x]_{+} + [\varphi_{-} - x]_{-}) + c^{-}[\varphi_{-} - x]_{-}. \end{aligned}$$

Proof From Figure 13 we see that

$$x \leq \varphi_{+} \Rightarrow z(x) = \varphi_{+} - x,\ x \geq \varphi_{-} \Rightarrow z(x) = \varphi_{+} - x,$$

from which the results follow. The behavior of $W(x)$ is illustrated in Figure 14. □

Proof of Theorem 1 The proof is by backward induction on stages. Since $g(d,x)$ is convex in x by (12), it follows from Corollary 1 applied to (21) that in state $x = x(k_1, \cdots, k_{m-1})$ at stage $(k_1, \cdots, k_m)$, the optimal decisions are given by the threshold rules

$$s^{*}_{+}(k_1, \cdots, k_m) = [\varphi_{+}(k_1, \cdots, k_m) - x]_{+},\ \ s^{*}_{+}(k_1, \cdots, k_m) = [\varphi_{-}(k_1, \cdots, k_m) - x]_{-},$$

with thresholds (24). Furthermore $J^{*}(x, k_1, \cdots, k_m)$ is given by (28) and it is convex. The proof is completed by backward induction applying Corollary 1 to (22) for $j \leq m-1$. □

B Proof of Theorem 2

Using definition (32) we can rewrite (24) as

$$-f_m(\varphi_\pm(m)) = -T_m c^\pm(m),$$

which is the same as (31). We now prove by backwards induction that $f_j(x)$ defined by (34) also satisfies

$$f_j(x) = -\mathbb{E}\{\sum_{k_{j+1}=0}^{N_{j+1}-1} \nabla J^*(x,k_1,\cdots,k_{j+1}) \mid \mathcal{Y}(t(k_1,\cdots,k_j))\}, \quad j \le m-1. \tag{73}$$

Suppose (73) is true. Then (26) implies that

$$-f_j(\varphi_\pm(j)) = -T_j c^\pm(j),$$

which is (33), as required. Thus we need to prove (73).

Consider the case $j = m-1$. Applying Corollary 1 to (21) shows that

$$\begin{aligned}
\mathbb{E}_m\{\nabla J^*(x,k_1,\cdots,k_m)\} &= -T_m c^+(m)\mathbf{1}(x \le \varphi_+(m)) - T_m c^-(m)\mathbf{1}(\varphi_-(m) \le x) \\
&+ T_m \mathbb{E}_m\{\nabla g\}\mathbf{1}(\varphi_+(m) < x < \varphi_-(m)), \text{ so} \\
\mathbb{E}_{m-1}\{\nabla J^*(x,k_1,\cdots,k_m)\} &= -T_m c^+(m)P_{m-1}(x \le \varphi_+(m)) - T_m c^-(m)P_{m-1}(\varphi_-(m) \le x) \\
&+ T_m \mathbb{E}_{m-1}\{\nabla g\mathbf{1}(\varphi_+(m) < x < \varphi_-(m))\}. \text{ Hence} \\
\mathbb{E}_{m-1}\{\sum_{k_m=0}^{N_m-1} \nabla J^*(x,k_1,\cdots,k_m)\} &= \sum_{k_m=0}^{N_m-1} \big[-T_m c^+(m)P_{m-1}(x \le \varphi_+(m)) - T_m c^-(m)P_{m-1}(\varphi_-(m) \le x) \\
&+ T_m \mathbb{E}_{m-1}\{\nabla g\mathbf{1}(\varphi_+(m) < x < \varphi_-(m))\}\big] = -f_{m-1}(x),
\end{aligned}$$

proving (73) for $j = m-1$.

We now prove that (73) is true for j, assuming it is true for $j+1$. From (34) we can verify that

$$\begin{aligned}
f_j(x) &= \sum_{k_{j+1}=0}^{N_{j+1}-1} \big[T_{j+1}c^+(j+1)P_j(x \le \varphi_+(j+1)) + T_{j+1}c^-(j+1)P_j(x \ge \varphi_-(j+1)) \\
&+ \mathbb{E}_j\{f_{j+1}(x)\mathbf{1}(\varphi_+(j+1) < x < \varphi_-(j+1)\}\big].
\end{aligned}$$

Since (73) is true for $j+1$, this gives

$$\begin{aligned}
f_j(x) &= \sum_{k_{j+1}=0}^{N_{j+1}-1} \mathbb{E}_j\{T_{j+1}c^+(j+1)\mathbf{1}(x \le \varphi_+(j+1)) + T_{j+1}c^-(j+1)\mathbf{1}(x \ge \varphi_-(j+1)) \\
&- \sum_{k_{j+2}=0}^{N_{j+2}=0} \nabla J^*(x,k_1,\cdots,k_{j+2})\mathbf{1}(\varphi_+(j+1) < x < \varphi_-(j+1))\}.
\end{aligned} \tag{74}$$

On the other hand, applying Corollary 1 to (22) gives

$$\begin{aligned}
\nabla J^*(x,k_1,\cdots,k_{j+1}) &= -T_{j+1}c^+(j+1)\mathbf{1}(x \le \varphi_+(j+1)) - T_{j+1}c^-(j+1)\mathbf{1}(\varphi_-(j+1) \le x) \\
&+ \sum_{k_{j+2}=0}^{N_{j+2}-1} \mathbb{E}_{j+1}\{\nabla J^*(x,k_1,\cdots,k_{j+2}\}\mathbf{1}(\varphi_+(j+1) < x < \varphi_-(j+1)).
\end{aligned} \tag{75}$$

Substitution of (75) into (74) gives

$$f_j(x) = -\sum_{k_{j+1}=0}^{N_{j+1}-1} \mathbb{E}_j\{\nabla J^*(x,k_1,\cdots,k_{j+1})\},$$

which is the same as (73), as required. □

C Proof of Theorem 3

The proof imitates the proof of Theorem 1 with an application of Lemma 3. □

D Proof of Lemma 2

To simplify notation, we write $\mu(j)=\mu(k_1,\cdots,k_j)$, $\varepsilon(j)=\varepsilon(k_1,\cdots,k_j)$, and so on, and $\gamma^+=\gamma^+(k_1,\cdots,k_m)$. Since $g(d,x)=\gamma^+[d-x]_+$, $\nabla g(d,x)=-\gamma^+\mathbf{1}(d-x\geq 0)$, and so from (32),

$$f_m(x) = T_m\gamma^+P_m(d(m)-x\geq 0) = T_m\gamma^+P_m([\mu(m)-x]+\varepsilon(m)\geq 0).$$

By (44), under P_m the random variable $[\mu(m)-x]+\varepsilon(m)$ is Gaussian with mean $[\mu(m)-x]$ and variance $\sigma^2(m)$. Hence the function $\psi(k_1,\cdots,k_m,\cdot)$ defined by

$$\psi(k_1,\cdots,k_m,x-\mu(m)) = T_m\gamma^+P_m(d(m)-x\geq 0), \tag{76}$$

is deterministic, which establishes (45) and hence (46) for $j=m$.

We proceed to $j=m-1$. By (34),

$$\begin{aligned} f_{m-1}(x) = \sum_{k_m=0}^{N_m-1} \quad & \big[T_m c^+(m)P_{m-1}(x\leq\varphi_+(m)) + T_m c^-(m)P_{m-1}(x\geq\varphi_-(m)) \\ & +T_m\gamma^+P_{m-1}(d(m)\geq x,\varphi_+(m)<x<\varphi_-(m))\big] \end{aligned} \tag{77}$$

Since

$$\begin{aligned} \varphi_\pm(m) &= \mu(m)+\Delta_\pm(m) = \mu(m-1)+\Delta_\pm(m)+\varepsilon(m-1), \\ d(m) &= \mu(m)+\varepsilon(m) = \mu(m-1)+\varepsilon(m-1)+\varepsilon(m), \end{aligned}$$

we have

$$\begin{aligned} \{x\leq\varphi_+(m)\} &= \{\varepsilon(m-1)+[\mu(m-1)-x]\geq -\Delta_+(m)\}, \\ \{x\geq\varphi_-(m)\} &= \{\varepsilon(m-1)+[\mu(m-1)-x]\leq -\Delta_-(m)\}, \\ \{d(m)\geq x,\varphi_+(m)<x<\varphi_-(m)\} &= \{\varepsilon(m-1)+\varepsilon(m)+[\mu(m-1)-x]\geq 0\} \\ &\cap \ \{-\Delta_-(m)<\varepsilon(m-1)+[\mu(m-1)-x]\leq -\Delta_+(m)\}. \end{aligned} \tag{78}$$

Under distribution P_{m-1}, the random vector $(\varepsilon(m-1)+[\mu(m-1)-x], \varepsilon(m-1)+\varepsilon(m)+[\mu(m-1)-x])$ is Gaussian with mean $[\mu(m-1)-x]e$ where e is a vector of all 1's, and covariance matrix $Q\Sigma Q^T$ with $\Sigma =$ diag $\{\sigma^2(m-1), \sigma^2(m)\}$ and

$$Q = \left[\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right].$$

It follows that P_{m-1} of each of the three events is a function of $[\mu(m-1)-x]$. This proves (45) for $j = m-1$.

Suppose now that (45) and hence (46) hold for $j+l,\ l \geq 1$. We show that (45) holds for j. From (34) we see that $f_j(x)$ is a linear combination of probabilities (under P_j) of intersections of events of the form

$$\begin{array}{rcl} \{x \leq \varphi_+(j+l)\} & = & \{\varepsilon(j)+\cdots\varepsilon(j+l-1)+[\mu(j)-x] \geq -\Delta_+(j+l)\}, \\ \{x \geq \varphi_-(j+l)\} & = & \{\varepsilon(j)+\cdots\varepsilon(j+l-1)+[\mu(j)-x] \leq -\Delta_+(j+l)\}, \\ \{\varphi_+(j+l) < x < \varphi_-(j+l)\} & = & \{-\Delta_-(j+l) \leq \varepsilon(j)+\cdots+\varepsilon(j+l-1)+[\mu(j)-x] \\ & & \leq -\Delta_+(j+l)\}. \end{array} \tag{79}$$

Under the distribution P_j, the random vector

$$\big(\varepsilon(j)+[\mu(j)-x], \varepsilon(j)+\varepsilon(j+1)+[\mu(j)-x], \cdots, \varepsilon(j)+\cdots+\varepsilon(m)+[\mu(j)-x]]\big)$$

is Gaussian with mean $[\mu(j)-x]e$ where e is a vector of all 1's, and covariance matrix $Q\Sigma Q^T$ with $\Sigma =$ diag $\{\sigma^2(j), \cdots, \sigma^2(m)\}$ and

$$Q = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ & & \cdots & \\ 1 & 1 & \cdots & 1 \end{array} \right].$$

So each probability in (34) can be expressed as explicit integrals of this multivariate distribution, hence as explicit functions of $[\mu(j)-x]$. This proves (45) for j. □

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